Homework No. 03 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Friday, 2020 Jan 31, 9:00 AM, in class

0. Problems 1 and 7 are to be submitted for assessment. Rest are for practice.

0. Keywords: Analytic function, Cauchy-Riemann conditions, Cauchy's integral formula, Residue theorem, Laurant series.

1. (**30 points.**) Analytic functions are significantly constrained, in that they have to satisfy the Cauchy-Riemann conditions. These conditions are necessary (but not sufficient) for a function of a complex variable to be analytic (differentiable). Check if the following functions satisfy the Cauchy-Riemann conditions.

$$f(z) = z^3, \tag{1a}$$
$$f(z) = |z|^2, \tag{1b}$$
$$f(z) = e^{iz}, \tag{1c}$$
$$f(z) = \ln z. \tag{1d}$$

2. (**20 points.**) Check if the following functions satisfies the Cauchy-Riemann conditions.

$$f(z) = zz^*, \tag{2a}$$
$$f(z) = e^{z+iz}, \tag{2b}$$
$$f(z) = e^z + e^{iz}, \tag{2c}$$

3. (**20 points.**) Check if the function

$$f(z) = \frac{1}{z} \tag{3}$$

satisfies the Cauchy-Riemann conditions.

   (a) Verify that the Cauchy-Riemann conditions for this case are not well defined at $z = 0$, but are fine for $z \neq 0$.

   (b) Verify that

$$\frac{df}{dz} = -\frac{1}{z^2}, \qquad z \neq 0. \tag{4}$$

   (c) Determine the limiting value of the derivative as you approach $z = 0$ along the positive real line, and, then, when you approach along the negative real line. Repeat the analysis along the imaginary line. Repeat the analysis along the line $x = y$. Are these limits identical?

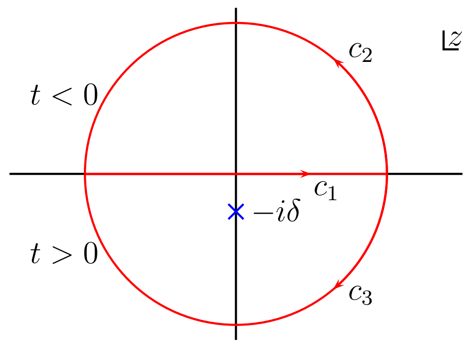

Figure 1: Contour $c = c_1 + c_2$ for $t < 0$ and the contour $c = c_1 + c_3$ for $t > 0$ used to evaluate the Heaviside step function.

(d) If these limits are not identical conclude that the derivative is not isotropic at $z = 0$. Then, the function is not analytic at $z = 0$.

4. (**10 points.**) Evaluate the following contour integrals. In the following the contour c is a unit circle going counterclockwise with center at $z = a$.

$$I(a) = \frac{1}{2\pi i} \oint_c dz \frac{(z^5 + 1)}{(z - a)}, \tag{5a}$$

$$I(a) = \frac{1}{2\pi i} \oint_c dz \frac{e^{iz}}{(z - a)}. \tag{5b}$$

5. (**20 points.**) Evaluate the contour integral

$$I = \frac{1}{2\pi i} \oint_c dz \frac{e^{iz}}{(z^2 - a^2)}, \tag{6}$$

where the contour c is a unit circle going counterclockwise with center at the origin. Inquire the cases when $|a| > 1$ and $|a| < 1$.

6. (**20 points.**) See Figure 1. Let $\delta > 0$.

(a) For $t < 0$ evaluate

$$G_1(t) = -\frac{1}{2\pi i} \oint_{c_1 + c_2} dz \, \frac{e^{-izt}}{z + i\delta}. \tag{7}$$

(b) For $t > 0$ evaluate

$$G_2(t) = -\frac{1}{2\pi i} \oint_{c_1 + c_3} dz \, \frac{e^{-izt}}{z + i\delta}, \tag{8}$$

where note that the contour is going clockwise, opposite of the convention used in Cauchy's theorem.

7. (**20 points.**) Consider the contour integral

$$I(a) = \frac{1}{2\pi i}\frac{2}{a}\oint_c \frac{dz}{\left(z^2 + \frac{2}{a}z + 1\right)}, \tag{9}$$

where the contour c is along the unit circle going counterclockwise. Show that

$$z^2 + \frac{2}{a}z + 1 = (z - r_+)(z - r_-), \tag{10}$$

where

$$r_\pm = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}. \tag{11}$$

Using residue theorem evaluate $I(\frac{1}{3})$.

8. (**20 points.**) Consider the contour integral

$$I(v, w) = \frac{1}{2\pi i}\frac{1}{2}\oint_c \frac{dz}{z}\frac{z^2 + 2\frac{w}{v}z + 1}{\left(z + \frac{v}{w}\right)\left(z + \frac{w}{v}\right)}, \tag{12}$$

where the contour c is along the unit circle going counterclockwise. Evaluate $I(1, 2)$ and $I(2, 1)$. In general, what happens when $v < w$ and $v > w$?