Homework No. 11 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

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Due date: Friday, 2020 Apr 17, 9:00 AM, in class

1. Keywords: Discrete Fourier transformation, Continuous Fourier transformation, Fourier series, Inverse Fourier tranform, Half-range Fourier series, Function space, Special functions, Orthogonality relations, Completeness relation.

2. Problem 1 (in Sec. 2.1) and Problem 1 (in Sec. 3.1) are to be submitted for assessment. Rest are lecture notes or problems for practice.

1 Vector space

1. A vector $\mathbf{A}$ in three dimensions can be expressed in the form

$$\mathbf{A} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3. \tag{1}$$

Here $\hat{\mathbf{e}}_i$ are called the basis vectors and a_i are components of the vector along the basis vectors.

(a) Orthogonality relation: Let us assume that the basis vectors are orthogonal to each other. This is stated compactly as

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{2}$$

where δ_{ij} is the Kronecker delta symbol.

(b) Vector components: Taking the dot product with $\hat{\mathbf{e}}_1$ in each term in Eq. (1) we obtain

$$\mathbf{A} \cdot \hat{\mathbf{e}}_1 = a_1(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1) + a_2(\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) + a_3(\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1). \tag{3}$$

Using the orthogonality relations between the basis vectors we immediately have

$$\mathbf{A} \cdot \hat{\mathbf{e}}_1 = a_1. \tag{4}$$

Similar relations can be derived for other components, and they can be together expressed in the form

$$\mathbf{A} \cdot \hat{\mathbf{e}}_i = a_i, \qquad i = 1, 2, 3. \tag{5}$$

(c) Completeness relation: Substituting the expressions for the vector components back in Eq. (1) we have

$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\mathbf{A}\cdot\hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 + (\mathbf{A}\cdot\hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3 \tag{6a}$$

$$= \mathbf{A}\cdot\Big[\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\Big], \tag{6b}$$

where the second equality is obtained by recognizing the common factor. Thus, the vector multiplied with the quantity inside square brackets returns back the vector. Since the multiplication involves a scalar dot product, the quantity in square brackets can not be a vector because then it will return a scalar. We identify it to be the unit dyadic. Thus,

$$\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 = \mathbf{1}, \tag{7}$$

which is the completeness relation for the basis vectors.

2 Discrete Fourier series

1. The Fourier space is spanned by the Fourier eigenfunctions

$$e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \ldots, \qquad 0 \le \phi < 2\pi. \tag{8}$$

An arbitrary function $f(\phi)$ has the Fourier series representation

$$f(\phi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} a_m e^{im\phi}, \tag{9}$$

where $e^{im\phi}$ are the Fourier eigenfunctions and a_m are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-in\phi}e^{im\phi} = \delta_{mn}. \tag{10}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$a_m = \int_0^{2\pi} d\phi\, e^{-im\phi} f(\phi). \tag{11}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im\phi}e^{-im\phi'} = \delta(\phi-\phi'). \tag{12}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{d\phi^2} - m^2\right] e^{im\phi} = 0. \tag{13}$$

(e) Green's function: The associated Green's function satisfies the equation

$$-\left[\frac{d^2}{d\phi^2} - m^2\right] g(\phi, \phi') = \delta(\phi - \phi'). \tag{14}$$

Verify by substitution that

$$g(\phi, \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{e^{in\phi} e^{-in\phi'}}{n^2 - m^2} \tag{15}$$

satisfies the Green function equation.

2.1 Problems

1. (**20 points.**) Determine all the Fourier components a_m for the following functions: $\cos\phi$, $\sin\phi$, $\cos^2\phi$, $\sin^2\phi$, $\cos^3\phi$, $\sin^3\phi$.

2. (**20 points.**) Determine the particular function $f(\phi)$ that has the Fourier components

$$a_m = 1 \tag{16}$$

for all m. That is, all the Fourier coefficients are contributing equally in the series.

3. (**20 points.**) To determine the Fourier components of $\tan\phi$ start from

$$\tan\phi = \frac{1}{i} \frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}} \tag{17}$$

and show that

$$\tan\phi = \frac{1}{i} + \sum_{m=1}^{\infty} e^{-2im\phi} \frac{2(-1)^m}{i}. \tag{18}$$

Thus, read out all the Fourier components. Similarly, find the Fourier components of $\cot\phi$.

4. (**20 points.**) Fourier series (or transformation) is defined as $(0 \leq \phi < 2\pi)$

$$f(\phi) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{im\phi} a_m, \tag{19}$$

where the coefficients a_m are determined using

$$a_m = \int_0^{2\pi} d\phi \, e^{-im\phi} f(\phi). \tag{20}$$

Determine all the Fourier components a_m for the function $\cos^3\phi$.

3 Continuous Fourier integral

1. The (continuous) Fourier space is spanned by the Fourier eigenfunctions

$$e^{ikx}, \qquad -\infty < k < \infty, \qquad -\infty < x < \infty. \tag{21}$$

An arbitrary function $f(x)$ has the Fourier series representation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{22}$$

where e^{ikx} are the Fourier eigenfunctions and $\tilde{f}(k)$ are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{-ik'x} e^{ikx} = \delta(k - k'). \tag{23}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x). \tag{24}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-ikx'} = \delta(x - x'). \tag{25}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{dx^2} - k^2\right] e^{ikx} = 0. \tag{26}$$

3.1 Problems

1. (**20 points.**) Fourier series (or transformation) is defined as $(-\infty < x < \infty)$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} a(k), \tag{27}$$

where the coefficients $a(k)$ are determined using

$$a(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x). \tag{28}$$

(a) Show that

$$\frac{d^n f(x)}{dx^n} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} (ik)^n e^{ikx} a(k). \tag{29}$$

(b) Show that the differential equation

$$-\left(\frac{d^2}{dx^2} - \omega^2\right) f(x) = \delta(x) \tag{30}$$

in the Fourier space is the algebraic equation

$$(k^2 + \omega^2) a(k) = 1. \tag{31}$$

Thus, the solution to the differential equation is the Fourier transform of

$$a(k) = \frac{1}{\omega^2 + k^2}. \tag{32}$$

2. (**20 points.**) Consider the inhomogeneous linear differential equation

$$\left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right) f(x) = \delta(x). \tag{33}$$

Use the Fourier transformation and the associated inverse Fourier transformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{34a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \tag{34b}$$

to show that the corresponding equation satisfied by $\tilde{f}(k)$ is algebraic. Find $\tilde{f}(k)$.

4 Half-range Fourier series

1. The half-range Fourier space is spanned by the Fourier eigenfunctions

$$\sin m\phi, \qquad m = 1, 2, 3, \ldots, \qquad 0 \leq \phi \leq \pi. \tag{35}$$

An arbitrary function $f(\phi)$, for ϕ limited to half the range, has the half-range Fourier series representation

$$f(\phi) = \sum_{m=1}^{\infty} a_m \sin m\phi, \tag{36}$$

where $\sin m\phi$ are the half-range Fourier eigenfunctions and a_m are the respective half-range Fourier components.

(a) Orthogonality relation: The half-range Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{2}{\pi} \int_0^{\pi} d\phi \, \sin m\phi \, \sin m'\phi = \delta_{mm'}. \tag{37}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$a_m = \frac{2}{\pi} \int_0^\pi d\phi \, \sin m\phi \, f(\phi). \tag{38}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\frac{2}{\pi} \sum_{m=1}^{\infty} \sin m\phi \, \sin m\phi' = \delta(\phi - \phi'). \tag{39}$$

(d) Differential equation: The half-range Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{d\phi^2} - m^2 \right] \sin m\phi = 0. \tag{40}$$

Note that half-range Fourier eigenfunctions are zero at $\phi = 0$ and $\phi = \pi$.

4.1 Problems

1. (**20 points.**) Prove the orthogonality relation

$$\frac{2}{\pi} \int_0^\pi d\phi \, \sin m\phi \, \sin m'\phi = \delta_{mm'}. \tag{41}$$

Hint: Use exponential representation for sin functions.

2. (**20 points.**) Prove the completeness relation

$$\frac{2}{\pi} \sum_{m=1}^{\infty} \sin m\phi \, \sin m\phi' = \delta(\phi - \phi'). \tag{42}$$

Note that ϕ and ϕ' are limited to the range 0 to π.
Hint: Use exponential representation for sin functions.

3. (**20 points.**) For ϕ limited to the range

$$0 \le \phi \le \pi \tag{43}$$

show that $\cos\phi$ can be expressed as a linear combination of sin functions. That is,

$$\cos\phi = \sum_{m=1}^{\infty} a_m \sin m\phi. \tag{44}$$

Show that

$$a_m = \begin{cases} 0, & m = 1, 3, 5, \ldots, \\ \frac{4}{\pi} \frac{m}{(m^2 - 1)}, & m = 2, 4, 6, \ldots. \end{cases} \tag{45}$$

Note that the series expansion is not valid at the boundaries $\phi = 0$ and $\phi = \pi$.

4. (**20 points.**) For ϕ limited to the range

$$0 \leq \phi \leq \pi \tag{46}$$

show that 1 can be expressed as a linear combination of sin functions. That is,

$$1 = \sum_{m=1}^{\infty} a_m \sin m\phi. \tag{47}$$

Show that

$$a_m = \begin{cases} \dfrac{4}{\pi} \dfrac{1}{m}, & m = 1, 3, 5, \ldots, \\ 0, & m = 2, 4, 6, \ldots. \end{cases} \tag{48}$$

Note that the series expansion is not valid at the boundaries $\phi = 0$ and $\phi = \pi$. Evaluate the series at $\phi = \pi/2$ and find the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots. \tag{49}$$