Homework No. 13 (2020 Spring)

PHYS 301: THEORETICAL METHODS IN PHYSICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Friday, 2020 May 1, 9:00 AM, in class

0. Keywords: Legendre polynomials, orthogonality relations, completeness relation.

0. Problem 2 and 4 are to be submitted for assessment. Rest are lecture notes or problems for practice.

1. (**20 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Note that $-1 \leq x \leq 1$. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$. In Mathematica these plots are generated using the following commands:
   `Plot[{LegendreP[0,x], LegendreP[1,x], LegendreP[2,x], LegendreP[3,x], LegendreP[4,x] },{x,-1,1}]`
   Compare your plots with those in Wikipedia article on 'Legendre Polynomials'. While there read the Wikipedia article on Adrien-Marie Legendre and the associated 'Portrait Debacle'.

2. (**20 points.**) Legendre polynomials are conveniently generated using the relation
$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{1}$$
   where $-1 \leq x \leq 1$. Evaluate Legendre polynomials of degree $l = 0, 1, 2, 3, 4$ in this manner.

3. (**20 points.**) Legendre polynomials $P_l(x)$ satisfy the relation
$$\int_{-1}^{1} dx\, P_l(x) = 0 \quad \text{for} \quad l \geq 1. \tag{2}$$
   Verify this explicitly for $l = 0, 1, 2, 3, 4$.

4. (**20 points.**) Legendre polynomials satisfy the differential equation
$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{3}$$
   Verify this explicitly for $l = 0, 1, 2, 3, 4$.

5. (**20 points.**) Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{4}$$

Verify this explicitly for $l = 0, 1, 2$ and $l' = 0, 1, 2$.

6. (**20 points.**) Legendre polynomials satisfy the completeness relation

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x) P_l(x') = \delta(x - x'). \tag{5}$$

This is for your information. No work needed.

7. (**Example.**) The Legendre polynomials of order l are

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{6}$$

In particular,

$$P_0(x) = 1, \tag{7a}$$
$$P_1(x) = x, \tag{7b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{7c}$$

The expansion

$$F(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} t^l P_l(x), \qquad |t| < 1, \tag{8}$$

is usually referred to as the generating function for Legendre's polynomials. From it all the properties of these polynomials may be derived.

8. (**Example.**) The Legendre polynomials of order l satisfy the recurrence relation

$$(2l+1) x P_l(x) = (l+1) P_{l+1}(x) + l P_{l-1}(x), \qquad l = 1, 2, 3, \ldots. \tag{9}$$

Recall,

$$P_0(x) = 1, \tag{10a}$$
$$P_1(x) = x. \tag{10b}$$

Derive the explicit expression for $P_l(x)$ for $l = 2, 3, 4, 5$ using the recurrence relation.