Midterm Exam No. 01 (2020 Spring)

PHYS 420: Electricity and Magnetism II

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1. (**20 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v}\times\mathbf{B}. \tag{1}$$

Choose $\mathbf{B}$ along the positive z-axis. The solution to this vector differential equation in terms of the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2a}$$
$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} - v_0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2b}$$

describes a circle. Determine the radius of the circle and the location of the center of the circle presuming the charge is positive. The detailed process of the derivation of the solution is optional for assessment.

2. (**20 points.**) (Based on Griffiths 4th ed. problem 5.45.)
A (hypothetical) stationary magnetic monopole q_m held fixed at the origin will have a magnetic field

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{\mathbf{r}}. \tag{3}$$

Oberve that $\boldsymbol{\nabla}\cdot\mathbf{B} \neq 0$ anymore. Consider the motion of another particle with mass m and electric charge q_e in the field of the magnetic monopole.

(a) Draw the magnetic field lines of the stationary magnetic monopole.

(b) The electric charge q_e experiences a force from the magnetic field given by

$$\mathbf{F} = q_e\mathbf{v}\times\mathbf{B}. \tag{4}$$

Thus, deduce the equation of motion for the electric charge to be

$$\frac{d\mathbf{v}}{dt} = \mathbf{v}\times\mathbf{r}\,\frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\frac{1}{m}, \tag{5}$$

where $\mathbf{v}$ is the velocity of the electric charge q_e.

(c) Recall that the motion of an electric charge in a uniform magnetic field implies circular (or helical) motion, which in turn implies that the speed $v = |\mathbf{v}|$ is a constant of motion. Show that the speed $v = |\mathbf{v}|$ is a constant of motion even for the motion of an electric charge moving in the field of a magnetic monopole. That is, show that

$$\frac{dv}{dt} = 0. \tag{6}$$

(Hint: Show that $v^2 = \mathbf{v} \cdot \mathbf{v}$ is a constant of motion. Use $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.)

3. (**20 points.**) A steady current I flows through a wire shown in Fig. 3. Find the magnitude and direction of magnetic field at point P. You are given the magnitude of the magnetic

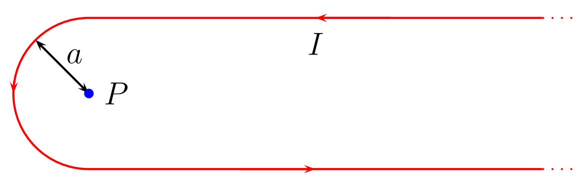

Figure 1: Problem 3.

field due to an infinite length of wire at distance ρ, and a circular loop of wire of radius R at the center of loop, to be

$$B_{\infty\text{-wire}} = \frac{\mu_0 I}{2\pi\rho} \qquad B_{\text{loop}} = \frac{\mu_0 I}{2R}. \tag{7}$$

4. (**20 points.**) A solenoid consists of a current carrying wire in the shape of a coil wound into a tightly packed helix on the surface of a circular cylinder of radius R. The magnetic field of a solenoid is characterized by the current I in the wire and the number of turns per unit length n of the coil. Remarkably, the magnetic field inside a solenoid is independent of the radius R. Using Ampere's law deduce the expression for the magnetic field inside and outside the solenoid.

5. (**20 points.**) The vector potential for a straight wire of infinite extent carrying a steady current I is

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}} \frac{\mu_0 I}{2\pi} \ln \frac{2L}{\rho}, \tag{8}$$

with $L \to \infty$ understood in the equation. The magnetic field around the wire is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi\rho}. \tag{9}$$

(a) Using an appropriate diagram describe the above vector potential and the magnetic field. Be precise.

(b) Evaluate $\boldsymbol{\nabla} \times \mathbf{A}$.

6. (**20 points.**) The current density for a straight wire of infinite extent carrying a steady current I_1 in the $\hat{\mathbf{z}}$ direction and passing through the origin is

$$\mathbf{j}_1(\mathbf{r}) = \hat{\mathbf{z}} I_1 \delta(x)\delta(y). \tag{10}$$

The magnetic vector potential generated by the stright wire is given by

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}} \frac{\mu_0}{4\pi} 2I_1 \ln \frac{2L}{\rho}, \tag{11}$$

where $\rho = \sqrt{x^2 + y^2}$ and $L \to \infty$ is understood in the equation. The magnetic field around the straight wire is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0}{4\pi} \frac{2I_1}{\rho}. \tag{12}$$

Let there be another straight wire of infinite extent carrying a steady current I_2 in the $\hat{\mathbf{z}}$ direction and passing through $x = a$ and $y = 0$ such that the two wires are parallel with separation distance a. The current density for the second wire is

$$\mathbf{j}_2(\mathbf{r}) = \hat{\mathbf{z}} I_2 \delta(x - a)\delta(y). \tag{13}$$

(a) The magnetostatic interaction energy of such a configuration of two wires is given by

$$W_{12} = -\int d^3r \, \mathbf{A}_1(\mathbf{r}) \cdot \mathbf{j}_2(\mathbf{r}), \tag{14}$$

where 1 in the subscript of $\mathbf{A}_1$ signifies that it is the magnetic vector potential due to the first wire. Similarly, $\mathbf{j}_2$ is the current density of the second wire. Find the expression for the interaction energy per unit length for the configuration of two parallel wires. Your expression for the interaction energy per unit length should be consistent with the experimental observation that 'like' currents attract and 'unlike' currents repel.

(b) Evaluate the expression for the force on one wire due to the other.