

Homework No. 03 (2020 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2019 Feb 3, 2:00 PM, in class

0. Keywords: Magnetostatics (Chap. 5, Griffiths 4th edition), Magnetic vector potential, Biot-Savart law (Sec. 5.2, Griffiths 4th edition).

0. Problems 2 and 4 are to be submitted for assessment. Rest are for practice.

1. **(20 points.)** The magnetic field $\mathbf{B}(\mathbf{r})$ is given in terms of the magnetic vector potential $\mathbf{A}(\mathbf{r})$ by the relation

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (1)$$

Find a magnetic vector potential (up to a gauge) for the uniform magnetic field

$$\mathbf{B} = B \hat{\mathbf{z}}. \quad (2)$$

Then, find another solution for \mathbf{A} (up to a gauge) that is different from your original solution by more than just a constant. If you designed an experiment to measure \mathbf{A} , which one of your solution will the experiment measure?

2. **(20 points.)** A homogeneous magnetic field \mathbf{B} is characterized by the vector potential

$$\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r}. \quad (3)$$

(a) Evaluate $\nabla \times \mathbf{A}$.

(b) Verify that this construction satisfies the radiation gauge by showing that

$$\nabla \cdot \mathbf{A} = 0. \quad (4)$$

(c) Is this construction unique?

3. **(20 points.)** Is it correct to conclude that

$$\nabla \cdot (\mathbf{r} \times \mathbf{A}) = -\mathbf{r} \cdot (\nabla \times \mathbf{A}), \quad (5)$$

where \mathbf{A} is a vector dependent on \mathbf{r} ? Explain your reasoning.

4. **(50 points.)** (Based on Problem 5.8, Griffiths 4th edition.)

The magnetic field at position $\mathbf{r} = (x, y, z)$ due to a finite wire segment of length $2L$

carrying a steady current I , with the caveat that it is unrealistic (why?), placed on the z -axis with its end points at $(0, 0, L)$ and $(0, 0, -L)$, is

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{4\pi} \frac{1}{\sqrt{x^2 + y^2}} \left[\frac{z + L}{\sqrt{x^2 + y^2 + (z + L)^2}} - \frac{z - L}{\sqrt{x^2 + y^2 + (z - L)^2}} \right], \quad (6)$$

where $\hat{\phi} = (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}) = (-y \hat{\mathbf{i}} + x \hat{\mathbf{j}}) / \sqrt{x^2 + y^2}$.

- (a) Show that by taking the limit $L \rightarrow \infty$ we obtain the magnetic field near a long straight wire carrying a steady current I ,

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{2\pi \rho}, \quad (7)$$

where $\rho = \sqrt{x^2 + y^2}$ is the perpendicular distance from the wire.

- (b) Show that the magnetic field on a line bisecting the wire segment is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{2\pi \rho} \frac{L}{\sqrt{\rho^2 + L^2}}. \quad (8)$$

- (c) Find the magnetic field at the center of a square loop, which carries a steady current I . Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2 + L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R} \frac{4}{\pi} \tan \frac{\pi}{4}. \quad (9)$$

- (d) Show that the magnetic field at the center of a regular n -sided polygon, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R} \frac{n}{\pi} \tan \frac{\pi}{n}, \quad (10)$$

where R is the distance from center to a corner of the polygon.

- (e) Show that the magnetic field at the center of a circular loop of radius R ,

$$B = \frac{\mu_0 I}{2R}, \quad (11)$$

is obtained in the limit $n \rightarrow \infty$.

5. **(20 points.)** The vector potential for a straight wire of infinite extent carrying a steady current I is

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}} \frac{\mu_0 I}{2\pi} \ln \frac{2L}{\rho}, \quad (12)$$

with $L \rightarrow \infty$ understood in the equation. The magnetic field around the wire is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{2\pi \rho}. \quad (13)$$

- (a) Draw the field lines for the above vector potential and the magnetic field. Be precise.
- (b) Evaluate $\nabla \times \mathbf{A}$.
6. **(20 points.)** The magnetic field for a straight wire of infinite extent carrying a steady current I is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\phi} \frac{\mu_0 I}{2\pi\rho}. \quad (14)$$

Verify that

$$\nabla \cdot \mathbf{B} = 0 \quad (15)$$

everywhere. In particular, investigate if the magnetic field is divergenceless on the wire, where $\rho = 0$. Next, evaluate

$$\nabla \times \mathbf{B} \quad (16)$$

everywhere. Thus, check if the magnetic field due to a straight current carrying wire satisfies the two Maxwell equations relevant for magnetostatics.