Homework No. 04 (2020 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Wednesday, 2019 Feb 12, 2:00 PM, in class

0. Keywords: Magnetic dipole moment, Rotating charged conductors.

0. Problems 1 and 2 are to be submitted for assessment. Rest are for practice.

1. (**20 points.**) Magnets are described by their magnetic moment. Estimate the magnetic moment of Earth (assuming it to be a point magnetic dipole $\mathbf{m}$. Next, similarly, estimate the magnetic moment of a typical refrigerator magnet.

2. (**20 points.**) A typical bar magnet is suitably approximated as a point magnetic dipole moment $\mathbf{m}$. The vector potential for a point magnetic dipole moment is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{1}$$

The magnetic field due to a point magnetic dipole $\mathbf{m}$ at a distance $\mathbf{r}$ away from the magnetic dipole is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\big[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3}, \qquad r \neq 0. \tag{2}$$

Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{m} = m\,\hat{\mathbf{z}}$.

(a) Qualitatively plot the magnetic field lines for the dipole $\mathbf{m}$. (Hint: You do not have to depend on Eq. (2) for this purpose. An intuitive knowledge of magnetic field lines should be the guide.)

(b) Find the expression for the magnetic field on the negative z-axis. (Hint: On the negative z-axis we have, $\hat{\mathbf{r}} = -\hat{\mathbf{z}}$ and $r = z$.)

3. (**20 points.**) The vector potential for a point magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{3}$$

Verify that the magnetic field due to the point dipole obtained by evaluating the curl

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{4}$$

can be expressed in the form, (using $\boldsymbol{\nabla}(1/r) = -\mathbf{r}/r^3$,)

$$\mathbf{B}(\mathbf{r}) = \mathbf{m}\,\mu_0\,\delta^{(3)}(\mathbf{r}) + \frac{\mu_0}{4\pi}(\mathbf{m}\cdot\boldsymbol{\nabla})\left(\boldsymbol{\nabla}\frac{1}{r}\right). \tag{5}$$

In this form it is easier to verify that the magnetic field satisfies the Maxwell equation

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{6}$$

Further, show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]}{r^3} + \mathbf{m}\,\mu_0\,\delta^{(3)}(\mathbf{r}). \tag{7}$$

This form, for regions outside the point dipole, brings out the dipole field,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]}{r^3}, \qquad r \neq 0. \tag{8}$$

4. (**20 points.**) The vector potential for a point magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{9}$$

Determine the corresponding magnetic field due to the point dipole using

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{10}$$

Find the simplified expression for the magnetic field everywhere along the line collinear to the magnetic moment $\mathbf{m}$. Next, find the simplified expression for the magnetic field in the plane containing the magnetic moment and perpendicular to the magnetic moment $\mathbf{m}$.

5. (**40 points.**) (Based on Problem 5.58, Griffiths 4th edition.) A circular loop of wire carries a charge q. It rotates with angular velocity $\boldsymbol{\omega}$ about its axis, say z-axis.

   (a) Show that the current density generated by this motion is given by

   $$\mathbf{J}(\mathbf{r}) = \frac{q}{2\pi a}\,\boldsymbol{\omega}\times\mathbf{r}\,\delta(\rho - a)\delta(z - 0). \tag{11}$$

   Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ for circular motion.

   (b) Using

   $$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{12}$$

   determine the magnetic dipole moment of this loop to be

   $$\mathbf{m} = \frac{qa^2}{2}\boldsymbol{\omega}. \tag{13}$$

(c) Calculate the angular momentum of the rotating loop to be

$$\mathbf{L} = ma^2\boldsymbol{\omega}, \tag{14}$$

where m is the mass of the loop.

(d) What is the gyromagnetic ratio g of the rotating loop, which is defined by the relation $\mathbf{m} = g\mathbf{L}$.