Homework No. 07 (2020 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

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Due date: Monday, 2019 Mar 2, 2:00 PM, in class

0. (**0 points.**) Keywords for finding resource materials: Electromagnetic energy conservation; Electromagnetic momentum conservation; Flux; Monochromatic electromagnetic wave.

0. Problems 3 is to be submitted for assessment. Rest are for practice.

1. (**60 points.**) When magnetic charges ρ_m and magnetic currents $\mathbf{j}_m$ are permitted, in addition to electric charges ρ_e and electric currents $\mathbf{j}_e$, the Maxwell equations are

$$\nabla \cdot \mathbf{D} = \rho_e, \tag{1a}$$

$$\nabla \cdot \mathbf{B} = \rho_m, \tag{1b}$$

$$-\nabla \times \mathbf{E} - \frac{\partial}{\partial t}\mathbf{B} = \mathbf{j}_m, \tag{1c}$$

$$\nabla \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{j}_e, \tag{1d}$$

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{2a}$$

$$\mathbf{B} = \mu_0 \mathbf{H}. \tag{2b}$$

The Lorentz force, in SI units, on an object with electric charge q_e and magnetic charge q_m is

$$\mathbf{F} = q_e\mathbf{E} + q_e\mathbf{v} \times \mathbf{B} + q_m\mathbf{H} - q_m\mathbf{v} \times \mathbf{D}. \tag{3}$$

Note the negative sign in the fourth term of the Lorentz force. This is postulated based on the observation that Maxwell equations are symmetric under the replacement

$$\rho_e \to \rho_m, \qquad \rho_m \to -\rho_e, \tag{4a}$$

$$\mathbf{j}_e \to \mathbf{j}_m, \qquad \mathbf{j}_m \to -\mathbf{j}_e, \tag{4b}$$

$$\mathbf{E} \to \mathbf{H}, \qquad \mathbf{H} \to -\mathbf{E}, \tag{4c}$$

which is a special case of duality transformation. The corresponding force density (force per unit volume) $\mathbf{f}$ is

$$\mathbf{f} = \rho_e\mathbf{E} + \mathbf{j}_e \times \mathbf{B} + \rho_m\mathbf{H} - \mathbf{j}_m \times \mathbf{D}. \tag{5}$$

The speed of light in vacuum c satisfies the relation

$$c^2\varepsilon_0\mu_0 = 1. \tag{6}$$

The duality transformation did entice us to consider magnetic monopoles. However, the purpose for introducing magnetic monopoles here is also to gain insight for the sources for the electromagnetic energy density and electromagnetic momentum density as suggested by the associated conservation laws deduced from the Maxwell equations. At any stage of our calculation we can get the standard electrodynamics by switching off the contributions from magnetic charges and currents, by setting $\rho_m = 0$ and $\mathbf{j}_m = 0$.

(a) Conservation of charge: Take the divergence of Ampère's law in Eq. (1d), and then use Gauss law for electric field in Eq. (1a) to deduce

$$\frac{\partial}{\partial t}\rho_e + \boldsymbol{\nabla}\cdot\mathbf{j}_e = 0. \tag{7}$$

This is the statement of conservation of electric charge. Similarly, take the divergence of Faraday's law in Eq. (1c), and then use Gauss law for magnetic field in Eq. (1b) to deduce

$$\frac{\partial}{\partial t}\rho_m + \boldsymbol{\nabla}\cdot\mathbf{j}_m = 0. \tag{8}$$

This is the statement of conservation of magnetic charge.

(b) Conservation of energy: The rate of energy transfer from the electromagnetic field to the charge, the power, is given by

$$\mathbf{F}\cdot\mathbf{v} = q_e\mathbf{v}\cdot\mathbf{E} + q_m\mathbf{v}\cdot\mathbf{H}. \tag{9}$$

For a continuous charge distribution, then, the rate of energy transfer from the electromagnetic field to charge distributions is given by

$$\mathbf{j}_e\cdot\mathbf{E} + \mathbf{j}_m\cdot\mathbf{H}. \tag{10}$$

Use Ampère's law in Eq. (1d) to replace $\mathbf{j}_e$, and Faraday's law in Eq. (1c) to replace $\mathbf{j}_m$, in Eq. (10), to obtain the statement of conservation of energy

$$\frac{\partial}{\partial t}U + \boldsymbol{\nabla}\cdot\mathbf{S} + \mathbf{j}_e\cdot\mathbf{E} + \mathbf{j}_m\cdot\mathbf{H} = 0, \tag{11}$$

where

$$U = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B} \tag{12}$$

is the electromagnetic field energy density and

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \tag{13}$$

is the flux of electromagnetic field energy density (the Poynting vector).
Hints: Use the identities

$$\mathbf{C}\cdot\frac{\partial}{\partial t}\mathbf{C} = \frac{\partial}{\partial t}\left(\frac{C^2}{2}\right) \tag{14}$$

for any vector $\mathbf{C}$, and

$$\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) = (\boldsymbol{\nabla}\times\mathbf{E})\cdot\mathbf{H} - (\boldsymbol{\nabla}\times\mathbf{H})\cdot\mathbf{E}. \tag{15}$$

(c) Conservation of momentum: We start from the expression for the force density in Eq. (5). Use Gauss law for electric and magnetic field in Eqs. (1a) and (1b) to replace ρ_e and ρ_m, Ampère's law in Eq. (1d) to replace $\mathbf{j}_e$, and Faraday's law in Eq. (1c) to replace $\mathbf{j}_m$, in Eq. (5), to obtain the statement of conservation of momentum

$$\frac{\partial}{\partial t}\mathbf{G} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0, \tag{16}$$

where

$$\mathbf{G} = \mathbf{D}\times\mathbf{B} \tag{17}$$

is the electromagnetic field momentum density and

$$\mathbf{T} = \mathbf{1}U - (\mathbf{ED} + \mathbf{HB}) \tag{18}$$

is the flux of electromagnetic field momentum density (the stress tensor).
Hints: Use the identities

$$(\boldsymbol{\nabla}\cdot\mathbf{D})\mathbf{E} + (\boldsymbol{\nabla}\times\mathbf{E})\times\mathbf{D} = -\frac{1}{2}\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{D}) + \boldsymbol{\nabla}\cdot(\mathbf{ED}), \tag{19a}$$

$$(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{H} + (\boldsymbol{\nabla}\times\mathbf{H})\times\mathbf{B} = -\frac{1}{2}\boldsymbol{\nabla}(\mathbf{H}\cdot\mathbf{B}) + \boldsymbol{\nabla}\cdot(\mathbf{HB}). \tag{19b}$$

2. (**20 points.**) Consider a uniformly charged spherical ball of radius R with total charge q.

(a) Using Gauss's law show that the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}}{R^3}, & r < R, \\ \dfrac{q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}}{r^3}, & R < r. \end{cases} \tag{20}$$

The magnetic field $\mathbf{B} = 0$ everywhere.

(b) Starting from the equation for conservation of electromagnetic linear momentum we have

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0. \tag{21}$$

Show that $\mathbf{G} = 0$. Thus, infer

$$\mathbf{f}\cdot\hat{\mathbf{r}} = -(\boldsymbol{\nabla}\cdot\mathbf{T})\cdot\hat{\mathbf{r}}. \tag{22}$$

(c) Consider a spherical volume V of radius r with the charge at the center. Note that $\mathbf{F} = \int d^3r\,\mathbf{f}$ will be zero due to spherical symmetry. To determine the electromagnetic stress (or the pressure, force per unit area,) on the sphere due to electrostatic repulsion between the constituent charges we define $F = \int_V d^3r\,(\mathbf{f}\cdot\hat{\mathbf{r}})$, which is the total sum of radial component of forces exerted on all the charges inside volume V by the electric and magnetic fields. The interpretation of this quantity as the force contributing to the pressure on the charge is brought out after we use divergence theorem to learn

$$F = -\oint_V d\mathbf{a}\cdot\mathbf{T}\cdot\hat{\mathbf{r}} = -\oint_V da\,\hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}}, \tag{23}$$

where we used $d\mathbf{a} = da\,\hat{\mathbf{r}}$. That is, F is the radial force on the charge due to the flux of electromagnetic momentum across the surface enclosing volume V. Note that $F \neq \mathbf{F}\cdot\hat{\mathbf{r}}$, because $\mathbf{F} = 0$.

(d) Evaluate

$$\hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}} = \begin{cases} -\dfrac{1}{8\pi}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{1}{r^4}, & r < R, \\ -\dfrac{1}{8\pi}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{r^2}{R^6}, & R < r. \end{cases} \tag{24}$$

(e) Thus, calculate the radial force F on the surface of the charge to be

$$F = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}. \tag{25}$$

Then, calculate the electromagnetic stress, F/area, on the surface of the charged sphere.

3. (**20 points.**) A monochromatic plane electromagnetic wave is described by electric and magnetic fields of the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{26a}$$

$$\mathbf{B} = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{26b}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constants. Assume no charges or currents.

(a) Using Maxwell's equations show that

$$\mathbf{k}\cdot\mathbf{E} = 0, \tag{27a}$$

$$\mathbf{k}\cdot\mathbf{B} = 0, \tag{27b}$$

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}, \tag{27c}$$

$$\mathbf{k}\times\mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}, \tag{27d}$$

where $\varepsilon_0\mu_0 = 1/c^2$.

(b) For non-trivial cases ($\mathbf{E}_0 \neq 0$ and $\mathbf{B}_0 \neq 0$), using Eqs. (27), show that we have

$$ck = \omega. \tag{28}$$

Then, deduce the relations

$$\mathbf{E}^*\cdot\mathbf{B} = 0, \tag{29}$$

$$\mathbf{E}^*\times\mathbf{B} = \hat{\mathbf{k}}\frac{1}{c}|\mathbf{E}|^2 = \hat{\mathbf{k}}c|\mathbf{B}|^2. \tag{30}$$

Thus, we have

$$E = cB. \tag{31}$$

(c) Evaluate the electromagnetic energy density

$$U = \frac{1}{2}\mathbf{D}^* \cdot \mathbf{E} + \frac{1}{2}\mathbf{B}^* \cdot \mathbf{H} \tag{32}$$

and the electromagnetic momentum density

$$\mathbf{G} = \mathbf{D}^* \times \mathbf{B}. \tag{33}$$

Then, determine the ratio U/G.