

Homework No. 09 (2020 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2019 Apr 6, 2:00 PM, in class

0. **(0 points.)** Keywords for finding resource materials: Relativistic kinematics, four-velocity, four-momentum, four-acceleration, proper acceleration, four-force.
0. Problems 2 is to be submitted for assessment. Rest are for practice.
1. **(100 points.)** Relativistic kinematics is constructed in terms of the proper time element ds , which remains unchanged under a Lorentz transformation,

$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}. \quad (1)$$

Here \mathbf{x} and t are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector

$$x^\alpha = (ct, \mathbf{x}). \quad (2)$$

(a) Velocity: The four-vector associated with velocity is constructed as

$$u^\alpha = c \frac{dx^\alpha}{ds}. \quad (3)$$

i. Using Eq. (1) deduce

$$\gamma ds = c dt, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \quad (4)$$

Then, show that

$$u^\alpha = (c\gamma, \mathbf{v}\gamma). \quad (5)$$

Here \mathbf{v} is the velocity that we use in Newtonian physics.

ii. Further, show that

$$u^\alpha u_\alpha = -c^2. \quad (6)$$

Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?

iii. Write down the form of the velocity four-vector in the rest frame of the particle?

(b) Momentum: Define momentum four-vector in terms of the mass m of the particle as

$$p^\alpha = mu^\alpha = (mc\gamma, m\mathbf{v}\gamma). \quad (7)$$

Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p} \right), \quad (8)$$

which corresponds to the definitions

$$E = mc^2\gamma, \quad (9a)$$

$$\mathbf{p} = m\mathbf{v}\gamma, \quad (9b)$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. Evaluate

$$p^\alpha p_\alpha = -m^2 c^2. \quad (10)$$

Thus, derive the energy-momentum relation

$$E^2 - p^2 c^2 = m^2 c^4. \quad (11)$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c \frac{du^\alpha}{ds}. \quad (12)$$

i. Show that

$$a^\alpha = \gamma \left(c \frac{d\gamma}{dt}, \mathbf{v} \frac{d\gamma}{dt} + \gamma \mathbf{a} \right), \quad (13)$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \quad (14)$$

is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (6) and taking derivative with respect to proper time show that

$$u^\alpha a_\alpha = 0. \quad (15)$$

Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (15) derive the identity

$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \right) \gamma^3. \quad (16)$$

iv. Show that

$$a^\alpha = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c} \gamma^4, \mathbf{a} \gamma^2 + \frac{\mathbf{v} \mathbf{v} \cdot \mathbf{a}}{c} \gamma^4 \right) \quad (17)$$

- v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where

$$\mathbf{a}_0 = \mathbf{a}|_{\text{rest frame}} \quad (18)$$

is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

- vi. Evaluate the following identities involving the proper acceleration

$$a^\alpha a_\alpha = \mathbf{a}_0 \cdot \mathbf{a}_0 = \left[\mathbf{a} \cdot \mathbf{a} + \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c} \right)^2 \gamma^2 \right] \gamma^4 = \left[\mathbf{a} \cdot \mathbf{a} - \left(\frac{\mathbf{v} \times \mathbf{a}}{c} \right)^2 \right] \gamma^6. \quad (19)$$

- vii. In a particular frame, if $\mathbf{v} \parallel \mathbf{a}$ (corresponding to linear motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}| \gamma^3. \quad (20)$$

And, in a particular frame, if $\mathbf{v} \perp \mathbf{a}$ (corresponding to circular motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}| \gamma^2. \quad (21)$$

- (d) Force: The force four-vector is defined as

$$f^\alpha = c \frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c} \frac{dE}{dt}, \mathbf{F} \gamma \right), \quad (22)$$

where the force \mathbf{F} , identified (or defined) as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad (23)$$

is the force in Newtonian physics. Starting from Eq. (10) derive the relation

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} \quad (24)$$

which is the power output or the rate of work done by the force \mathbf{F} on the particle.

- (e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = m a^\alpha. \quad (25)$$

Show that these involve the relations

$$\mathbf{F} = m \mathbf{a} \gamma + m \mathbf{v} \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \gamma^3, \quad (26a)$$

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} = m \mathbf{v} \cdot \mathbf{a} \gamma^3. \quad (26b)$$

2. **(20 points.)** Lorentz transformation relates the energy of momentum of a particle when measured in different frames. For example, for the special case when the relative velocity and the velocity of the particle are parallel we have

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}. \quad (27)$$

Photons are massless spin 1 particles whose energy and momentum are $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, such that $\omega = kc$. Thus, derive the relativistic Doppler effect formula

$$\omega' = \omega \sqrt{\frac{1 + \beta}{1 - \beta}}. \quad (28)$$