Homework No. 12 (2020 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Friday, 2020 May 1, 2:00 PM, in class

0. (**0 points.**) Keywords for finding resource materials: Rayleigh scattering.

1. (**20 points.**) The scattering boundary conditions imposes the far-field approximation $r' \ll r$, which amounts to the replacement

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'} = r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) + \mathcal{O}\left(\frac{r'}{r}\right)^2. \tag{1}$$

Show that in the far-field asymptotic limit we can replace

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \to \frac{e^{ikr}}{4\pi r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \tag{2}$$

where we introduced the notation

$$\mathbf{k}' = k\,\hat{\mathbf{r}} \tag{3}$$

with $k = 2\pi/\lambda$ given in terms of wavelength λ. In this form we see the structure of the spherical outgoing wave e^{ikr}/r. Show that the far-field approximation allows the replacement

$$\boldsymbol{\nabla}\frac{e^{ikr}}{r} \to i\mathbf{k}'\,\frac{e^{ikr}}{r}. \tag{4}$$

Show that these lead to the dyadic transcription

$$-\left(\boldsymbol{\nabla}\boldsymbol{\nabla} + \frac{\omega^2}{c^2}\mathbf{1}\right)\frac{e^{ikr}}{r} = \left(\mathbf{k}'\mathbf{k}' - \frac{\omega^2}{c^2}\mathbf{1}\right)\frac{e^{ikr}}{r} = \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})k^2\frac{e^{ikr}}{r}, \tag{5}$$

where $\mathbf{1}$ is the unit dyadic.