Midterm Exam No. 01 (2020 Spring)

PHYS 510: CLASSICAL MECHANICS

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1. (**20 points.**) Given the functional

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) \frac{du(x)}{dx}. \tag{1}$$

Assuming no variations at the end points, evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)}. \tag{2}$$

Here $a(x)$ is a known function.

2. (**20 points.**) Prove the intuitively obvious statement that the curve of shortest distance going through two points on a plane, the geodesics of a plane, are straight lines passing through the two points.

3. (**20 points.**) A free relativistic particle of mass m is described by

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = 0, \tag{3}$$

where c is speed of light in vacuum. Find a Lagrangian for this system that implies the equation of motion in Eq. (3).

4. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless rigid rod of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod such that the coordinates of mass m_1 are

$$x_1(t) = A \sin \omega_F t, \tag{4}$$
$$y_1(t) = 0. \tag{5}$$

See Figure 1. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod. Determine the equations of motion for the mass m_2 and qualitatively interpret the motion.

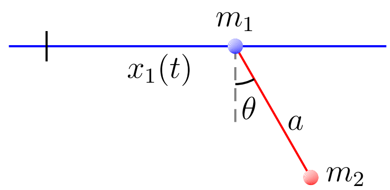

Figure 1: Problem 4.

5. (**20 points.**) Fermat's principle in ray optics states that a ray of light takes the path of least time between two given points. The speed of light in a medium is given in terms of the refractive index

$$n = \frac{c}{v}, \tag{6}$$

of the medium, where c is the speed of light in vacuum and v is the speed of light in the medium. Consider a ray of light traversing a path from (x_1, y_1) to (x_2, y_2) in a plane of fixed z.

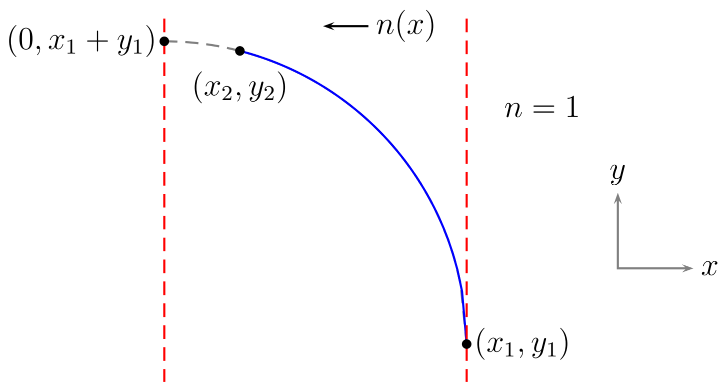

Figure 2: Problem 5.

(a) Show that the time taken to travel an infinitesimal distance ds is given by

$$dt = \frac{ds}{v} = \frac{n\,ds}{c}, \tag{7}$$

where ds in a plane is characterized by the infinitesimal statement

$$ds^2 = dx^2 + dy^2. \tag{8}$$

(b) Fermat's principle states that the path traversed by a ray of light from (x_1, y_1) to (x_2, y_2) is the extremal of the functional

$$T[y] = \frac{1}{c}\int_{(x_1,y_1)}^{(x_2,y_2)} n\,ds = \frac{1}{c}\int_{x_1}^{x_2} dx\, n(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{9}$$

(c) Since the ray of light passes through the points (x_1, y_1) and (x_2, y_2), we do not consider variations at these (end) points. Thus, show that

$$\frac{\delta T[y]}{\delta y(x)} = -\frac{d}{dx}\left[\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\right]. \tag{10}$$

(d) Using Fermat's principle show that the differential equation for the path $y(x)$ traversed by the ray of light is

$$\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = n_0, \tag{11}$$

where n_0 is a constant. Show that the above equation can be rewritten in the form

$$\frac{dy}{dx} = \frac{n_0}{\sqrt{n(x)^2 - n_0^2}}. \tag{12}$$

(e) Let us consider a medium with refractive index $(x_1 = a)$

$$n(x) = \begin{cases} \dfrac{a}{x}, & 0 < x < a, \\ 1, & a < x. \end{cases} \tag{13}$$

Solve the corresponding differential equation to obtain

$$y(x) - y_0 = \frac{1}{n_0}\left[\sqrt{a^2 - n_0^2 x^2} - \sqrt{a^2 - n_0^2 a^2}\right], \qquad x < a. \tag{14}$$

The path in this medium satisfies the equation of a circle. Determine the radius of the circle to be a/n_0 and the location of the center to be $(0, y_0 - a\sqrt{(1/n_0^2) - 1})$. For initial conditions

$$y(x_1) = y_1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} = y_1' \tag{15}$$

show that the integration constants are determined to be

$$y_0 = y_1 \qquad \text{and} \qquad n_0 = \frac{y_1'}{\sqrt{1 + {y_1'}^2}}. \tag{16}$$

For the special case when $y_1 = 0$ and $y_1' \to \infty$ show that $n_0 = 1$ and

$$y(x) = \sqrt{a^2 - x^2}, \qquad x < a. \tag{17}$$

Evaluate the total time taken for light to go from $(x_1 = a, y_1 = 0)$ to $(x_2 = 0, y_2 = a)$.