Midterm Exam No. 02 (2020 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

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1. (**20 points.**) A general rotation in 3-dimensions can be written in terms of consecutive rotations about x, y, and z axes,

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{1}$$

For infinitesimal rotations use

$$\cos\theta_i \sim 1, \tag{2a}$$

$$\sin\theta_i \sim \theta_i \to \delta\theta_i, \tag{2b}$$

to obtain

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{3}$$

Show that this leads to the vector relation

$$\mathbf{r}' = \mathbf{r} - \delta\boldsymbol{\theta} \times \mathbf{r}, \tag{4}$$

where

$$\mathbf{r} = x_1\hat{\mathbf{x}} + x_2\hat{\mathbf{y}} + x_3\hat{\mathbf{z}}, \tag{5a}$$

$$\delta\boldsymbol{\theta} = \delta\theta_1\hat{\mathbf{x}} + \delta\theta_2\hat{\mathbf{y}} + \delta\theta_3\hat{\mathbf{z}}. \tag{5b}$$

As a particular example, verify that a rotation about the direction $\hat{\mathbf{z}}$ by an infinitesimal (azimuth) angle $\delta\phi$ is described by

$$\delta\boldsymbol{\theta} = \hat{\mathbf{z}}\,\delta\phi. \tag{6}$$

The corresponding infinitesimal transformation in $\mathbf{r}$ is given by

$$\delta\mathbf{r} = \delta\phi\,\hat{\mathbf{z}} \times \mathbf{r} = \hat{\boldsymbol{\phi}}\,\rho\delta\phi, \tag{7}$$

where ρ and ϕ are the cylindrical coordinates defined as

$$\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \quad \text{and} \quad |\hat{\mathbf{z}} \times \mathbf{r}| = \rho. \tag{8}$$

Observe that, in rectangular coordinates $\rho\hat{\boldsymbol{\phi}} = x\hat{\mathbf{y}} - y\hat{\mathbf{x}}$.

2. (**20 points.**) Consider two discs of radii r_1 and r_2, and moment of inertia I_1 and I_2. Disc 1 is free to roll about an axis parallel to z axis passing through its center O_1. Similarly, disc 2 is free to roll about an axis parallel to z axis passing through its center O_2. Further, the center of disc 2 is free to move on a circle of radii $(r_1 + r_2)$. See Figure 2. Assume gravity in the direction of z axis and no motion in the z direction. The two discs are in contact with sufficient friction between them such that the resultant motion leads to perfect rolling of the surfaces,

$$\theta_1 r_1 = \theta_2 r_2. \tag{9}$$

Here θ_1 and θ_2 are angular displacements of the respective discs about the axes O_1 and O_2. The angular displacement of the axis O_2 about the axis O_1 is parametrized by the angular displacement α_2. Assume the discs are rolling under the action of no external

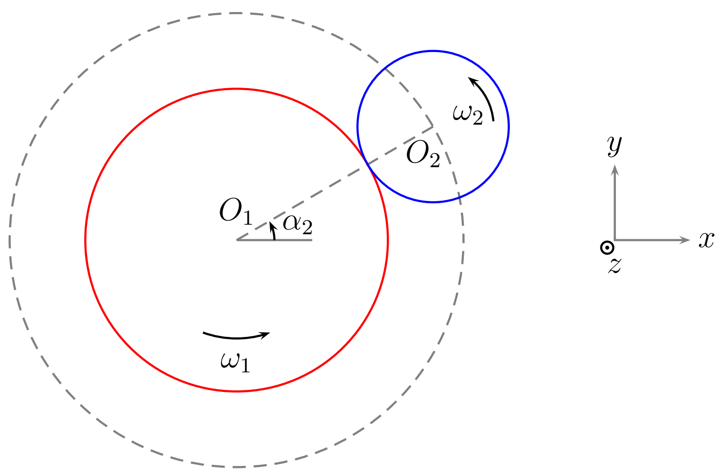

Figure 1: Problem 2.

torques. Write the Lagrangian for this system in terms of the coordinates θ_1, θ_2, and α_2, and their derivatives,

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \alpha_2, \dot{\alpha}_2). \tag{10}$$

Find the equations of motion and describe the motion.

3. (**20 points.**) In the Kepler problem the orbit of a planet is a conic section

$$r(\phi) = \frac{r_0}{1 - e\cos\phi} \tag{11}$$

expressed in terms of the eccentricity e and distance r_0. The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \tag{12}$$

is minimum at r_0. We used the definitions

$$r_0 = \frac{L_z^2}{\mu\alpha}, \qquad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \qquad e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \tag{13}$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z, which are constants of motion. The statement of conservation of angular momentum can be expressed in the form

$$dt = \frac{\mu}{L_z} r^2 d\phi, \tag{14}$$

which is convenient for evaluating the time elapsed in the motion. For the case of elliptic orbit, $U_{\text{eff}}(r_0) < E < 0$, show that the time period is given by

$$T = \frac{\mu}{L_z}\int_0^{2\pi} d\phi \frac{r_0^2}{(1 - e\cos\phi)^2} = \frac{\mu r_0^2}{L_z}\frac{2\pi}{(1-e^2)^{\frac{3}{2}}}. \tag{15}$$

Show that at point '2' in Figure 3

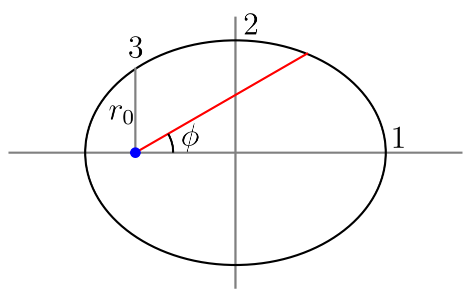

Figure 2: Elliptic orbit

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right), \quad \text{and} \quad r = a. \tag{16}$$

The time taken to go from '1' to '2' is given by (need not be proved here)

$$t_{1\to 2} = \frac{\mu}{L_z}\int_0^{\tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)} d\phi \frac{r_0^2}{(1 - e\cos\phi)^2} = \left(\frac{\pi}{2} + e\right)\frac{T}{2\pi}. \tag{17}$$

Show that at point '3' in Figure 3

$$\phi = \frac{\pi}{2}, \quad \text{and} \quad r = r_0. \tag{18}$$

The time taken to go from '1' to '3' is given by (need not be proved here)

$$t_{1\to 3} = \frac{\mu}{L_z}\int_0^{\frac{\pi}{2}} d\phi \frac{r_0^2}{(1-e\cos\phi)^2} = \left(e\sqrt{1-e^2} + 2\sin^{-1}\sqrt{\frac{1+e}{2}}\right)\frac{T}{2\pi}. \tag{19}$$

The eccentricity e of Earth's orbit is 0.0167 and timeperiod T is 365 days. Thus, calculate

$$t_{1\to 3} - t_{1\to 2} \tag{20}$$

for Earth in units of days.

4. (**20 points.**) Refer to the essay by J. M. Luttinger titled 'On "negative" mass in the theory of gravitation' in 1951.
   (a) Reproduce all the equations in the essay.
   (b) Critically assess the logic of the arguments in the essay.