Homework No. 02 (2020 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2020 Feb 4, 4.30pm

1. (**60 points.**) Fermat's principle in ray optics states that a ray of light takes the path of least time between two given points. The speed of light in a medium is given in terms of the refractive index

$$n = \frac{c}{v}, \tag{1}$$

of the medium, where c is the speed of light in vacuum and v is the speed of light in the medium. Consider a ray of light traversing a path from (x_1, y_1) to (x_2, y_2) in a plane of fixed z.

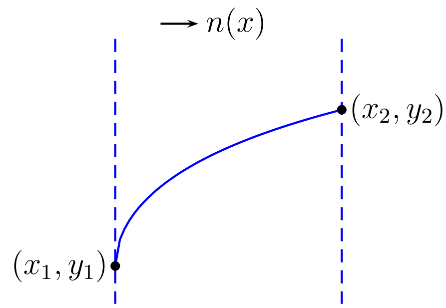

Figure 1: Problem 1.

(a) Show that the time taken to travel an infinitesimal distance ds is given by

$$dt = \frac{ds}{v} = \frac{n\,ds}{c}, \tag{2}$$

where ds in a plane is characterized by the infinitesimal statement

$$ds^2 = dx^2 + dy^2. \tag{3}$$

(b) Fermat's principle states that the path traversed by a ray of light from (x_1, y_1) to (x_2, y_2) is the extremal of the functional

$$T[y] = \frac{1}{c}\int_{(x_1,y_1)}^{(x_2,y_2)} n\,ds = \frac{1}{c}\int_{x_1}^{x_2} dx\, n(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4}$$

(c) Since the ray of light passes through the points (x_1, y_1) and (x_2, y_2), we do not consider variations at these (end) points. Thus, show that

$$\frac{\delta T[y]}{\delta y(x)} = -\frac{d}{dx}\left[\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\right]. \tag{5}$$

(d) Using Fermat's principle show that the differential equation for the path $y(x)$ traversed by the ray of light is

$$\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = n_1, \tag{6}$$

where n_1 is a constant. Show that the above equation can be rewritten in the form

$$\frac{dy}{dx} = \frac{n_1}{\sqrt{n(x)^2 - n_1^2}}. \tag{7}$$

(e) Let us consider a medium with refractive index

$$n(x) = \begin{cases} 1, & x < x_1, \\ \dfrac{x}{x_1}, & x > x_1. \end{cases} \tag{8}$$

Further, let

$$y(x_1) = y_1 = 0, \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} \to \infty. \tag{9}$$

Show that for this case $n_1 = 1$. Solve the corresponding differential equation to obtain

$$y(x) = x_1 \ln\left[\frac{x}{x_1} + \sqrt{\left(\frac{x}{x_1}\right)^2 - 1}\right], \qquad x_1 < x. \tag{10}$$

Show that the path satisfies the equation of a catenary

$$\cosh\frac{y}{x_1} = \frac{x}{x_1}. \tag{11}$$

Evaluate the total time taken for the light to go from (x_1, y_1) to (x_2, y_2).