

# Homework No. 05 (2020 Spring)

## PHYS 510: CLASSICAL MECHANICS

*Department of Physics, Southern Illinois University–Carbondale*

Due date: Tuesday, 2020 Mar 3, 4.30pm

1. **(20 points.)** A pendulum consists of a mass  $m_2$  hanging from a pivot by a massless string of length  $a_2$ . The pivot, in general, has mass  $m_1$ , but, for simplification let  $m_1 = 0$ . Let the pivot be constrained to move on a frictionless hoop of radius  $a_1$ . See Figure 1. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.

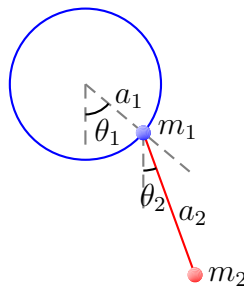


Figure 1: Problem 1.

- (a) Determine the Lagrangian for the system to be

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{1}{2}m_2a_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2a_2^2\dot{\theta}_2^2 + m_2a_1a_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2ga_1 \cos \theta_1 + m_2ga_2 \cos \theta_2. \quad (1)$$

- (b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_2a_1^2\dot{\theta}_1 + m_2a_1a_2\dot{\theta}_2 \cos(\theta_1 - \theta_2), \quad (2a)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2a_1a_2\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2ga_1 \sin \theta_1, \quad (2b)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2a_2^2\dot{\theta}_2 + m_2a_1a_2\dot{\theta}_1 \cos(\theta_1 - \theta_2), \quad (2c)$$

$$\frac{\partial L}{\partial \theta_2} = m_2a_1a_2\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2ga_2 \sin \theta_2. \quad (2d)$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{\theta}_1 + \omega_1^2 \sin \theta_1 + \frac{1}{\beta} \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta} \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \quad (3a)$$

$$\ddot{\theta}_2 + \omega_2^2 \sin \theta_2 + \beta \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \beta \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0, \quad (3b)$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \quad (4)$$

Note that  $\beta$  is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass  $m_2$  when  $m_1 = 0$ .

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2 \theta_1 + \frac{1}{\beta} \ddot{\theta}_2 = 0, \quad (5a)$$

$$\ddot{\theta}_2 + \omega_2^2 \theta_2 + \beta \ddot{\theta}_1 = 0. \quad (5b)$$

(e) Determine the solution for the initial conditions

$$\theta_1(0) = \theta_2(0) = \theta_{20}, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0. \quad (6)$$

Interpret and expound your solution.

2. (20 points.) Consider the coplanar double pendulum in Figure 2.

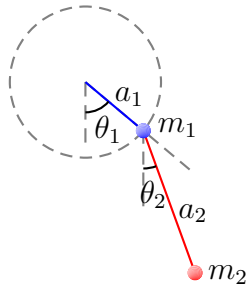


Figure 2: Problem 2.

(a) Write the Lagrangian for the system. in particular, show that the Lagrangian can be expressed in the form

$$L = L_1 + L_2 + L_{\text{int}}, \quad (7)$$

where

$$L_1 = \frac{1}{2}(m_1 + m_2)a_1^2\dot{\theta}_1^2 + (m_1 + m_2)ga_1 \cos \theta_1, \quad (8a)$$

$$L_2 = \frac{1}{2}m_2a_2^2\dot{\theta}_2^2 + m_2ga_2 \cos \theta_2, \quad (8b)$$

$$L_{\text{int}} = m_2a_1a_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2). \quad (8c)$$

(b) Determine the equations of motion for the system. Express them in the form

$$(m_1 + m_2)a_1\ddot{\theta}_1 + (m_1 + m_2)g \sin \theta_1 + m_2a_2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2a_2\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \quad (9a)$$

$$a_2\ddot{\theta}_2 + g \sin \theta_2 + a_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - a_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0. \quad (9b)$$

(c) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \frac{\alpha}{\beta}\ddot{\theta}_2 = 0, \quad (10a)$$

$$\ddot{\theta}_2 + \omega_2^2\theta_2 + \beta\ddot{\theta}_1 = 0, \quad (10b)$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \alpha = \frac{m_2}{m_1 + m_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \quad (11)$$

Note that  $0 \leq \alpha \leq 1$ .

(d) Determine the solution for the initial conditions

$$\theta_1(0) = 0, \quad \theta_2(0) = 0, \quad \dot{\theta}_1(0) = 0, \quad \dot{\theta}_2(0) = \omega_0, \quad (12)$$

for  $\alpha = 1/2$  and  $\beta = 1$ .