Homework No. 01 (2021 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Wednesday, 2021 Jan 27, 2:00 PM

0. Keywords: Motion of a charged particle in electric and magnetic field.

0. Problems 1, 2, and 4 are to be submitted for assessment. Rest are for practice.

1. (**30 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v}\times\mathbf{B}. \tag{1}$$

Choose $\mathbf{B}$ along the z-axis and solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2a}$$

$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + v_0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{2b}$$

Verify that the solution describes a circle of radius R with center at position $R\,\hat{\mathbf{i}}$. Find R. For the same initial velocity does an electron or a proton have a larger radii.

2. (**30 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ and a uniform electric field $\mathbf{E}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} + q\,\mathbf{v}\times\mathbf{B}. \tag{3}$$

Choose $\mathbf{B}$ along the z-axis and $\mathbf{E}$ along the y-axis,

$$\mathbf{B} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + B\,\hat{\mathbf{k}}, \tag{4a}$$

$$\mathbf{E} = 0\,\hat{\mathbf{i}} + E\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{4b}$$

Solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{5a}$$

$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{5b}$$

Verify that the solution is a cycloid characterized by the equations

$$x(t) = R(\omega_c t - \sin\omega_c t), \tag{6a}$$

$$y(t) = R(1 - \cos\omega_c t). \tag{6b}$$

where

$$R = \frac{E}{B\omega_c}, \qquad \omega_c = \frac{qB}{m}. \tag{7}$$

The particle moves as though it were a point on the rim of a wheel of radius R perfectly rolling (without sliding or slipping) with angular speed ω_c along the x-axis. It satisfies the equation of a circle of radius R whose center $(vt, R, 0)$ travels along the x-direction at constant speed v,

$$(x - vt)^2 + (y - R)^2 = R^2, \tag{8}$$

where $v = \omega_c R$.

3. (**20 points.**) (Based on Griffiths 4th ed. problem 5.45.)
A (hypothetical) stationary magnetic monopole q_m held fixed at the origin will have a magnetic field

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{\mathbf{r}}, \tag{9}$$

because $\boldsymbol{\nabla} \cdot \mathbf{B} \neq 0$ anymore. Consider the motion of a particle with mass m and electric charge q_e in the field of this magnetic monopole.

   (a) Draw the magnetic field lines of the stationary magnetic monopole.

   (b) Using

$$\mathbf{F} = q_e \mathbf{v} \times \mathbf{B} \tag{10}$$

   derive the equation of motion for the electric charge to be

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{r}\, \frac{\mu_0}{4\pi}\frac{q_e q_m}{r^3}\frac{1}{m}, \tag{11}$$

   where $\mathbf{v}$ is the velocity of the electric charge q_e.

   (c) Recall that the motion of an electric charge in a uniform magnetic field implies circular (or helical) motion, which in turn implies that the speed $v = |\mathbf{v}|$ is a constant of motion. Show that the speed $v = |\mathbf{v}|$ is a constant of motion even for the motion of an electric charge in the field of a magnetic monopole. That is, show that

$$\frac{dv}{dt} = 0. \tag{12}$$

   (Hint: Show that $v^2 = \mathbf{v}\cdot\mathbf{v}$ is a constant of motion. Use $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0$.) However, the motion is not circular. Nevertheless, it is exactly solvable and the orbit is unbounded and lies on a right circular half-cone with vertex at the monopole. The comments following Eq. (12) are for your information and need not be proved here.

4. (**20 points.**) The force $d\mathbf{F}$ on an infinitely small line element $d\mathbf{l}$ of wire, carrying steady current I, placed in a magnetic field $\mathbf{B}$, is

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}. \tag{13}$$

This involves the correspondence

$$q\mathbf{v} \to I d\mathbf{l} \tag{14}$$

for the flow of charge, representing current, in the wire. Consider a wire segment of arbitrary shape (in the shape of a curve C) with one end at the origin and the other end at the tip of vector $\mathbf{L}$. The total force on the segment of wire is given by the line integral

$$\mathbf{F} = \int_{\mathbf{0}\,(\text{path}\,C)}^{\mathbf{L}} I d\mathbf{l} \times \mathbf{B}. \tag{15}$$

Evaluate the total force on a closed loop of wire (of arbitrary shape and carrying steady current I) when it is placed in a uniform magnetic field? Check your result for a loop of wire in the shape of a square in a uniform magnetic field.