Homework No. 05 (2021 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2021 Mar 8, 2:00 PM

0. (**0 points.**) Keywords for finding resource materials: Complete elliptic integrals; Magnetic vector potential and magnetic field for a circular loop carrying a steady current.

0. Problems 3 and 4 are to be submitted for assessment. Rest are for practice.

1. (**0 points.**) Complete elliptic integrals of the first and second kind can be defined using the integral representations,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1 - k^2 \sin^2\psi}}, \tag{1a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1 - k^2 \sin^2\psi}, \tag{1b}$$

respectively.

2. (**20 points.**) Verify that

$$K(0) = \frac{\pi}{2}, \tag{2a}$$

$$E(0) = \frac{\pi}{2}. \tag{2b}$$

Then, verify that

$$E(1) = 1. \tag{3}$$

Note that

$$K(1) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\cos\psi} \tag{4}$$

is devergent. To see the nature of this divergence we introduce a cutoff parameter $\delta > 0$ and write

$$K(1) = \int_0^{\frac{\pi}{2}-\delta} \frac{d\psi}{\cos\psi}. \tag{5}$$

Evaluate the integral, (using the identity $d(\sec\psi + \tan\psi)/d\psi = \sec\psi(\sec\psi + \tan\psi)$,) and show that

$$K(1) \sim \ln 2 - \ln\delta - \frac{\delta^2}{12} + \mathcal{O}(\delta)^4 \tag{6}$$

has logarithmic divergence. Using Mathematica (or another graphing tool) plot $K(k)$ and $E(k)$ as functions of k for $0 \leq k < 1$.

3. (**20 points.**) The complete elliptic integrals have the power series expansions

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1+\frac{1}{4}k^2+\frac{9}{64}k^4+\ldots\right], \tag{7a}$$

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{(1-2n)} = \frac{\pi}{2}\left[1-\frac{1}{4}k^2-\frac{3}{64}k^4-\ldots\right]. \tag{7b}$$

The leading order contribution in the power series expansions are from $K(0)$ and $E(0)$. Evaluate the next-to-leading order contributions in the above series expansions by expanding the radical in Eqs.(1) as a series. Use

$$\frac{1}{\sqrt{1-x}} = 1+\frac{1}{2}x+\ldots, \tag{8a}$$

$$\sqrt{1-x} = 1-\frac{1}{2}x+\ldots. \tag{8b}$$

4. (**20 points.**) A circular loop of radius a carrying a steady current I with the loop chosen to be in the x-y plane with the origin at the center of the loop has the the magnetic vector potential given by

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi}\frac{4a}{\sqrt{z^2+(\rho+a)^2}}\left[\frac{2}{k^2}\Big\{K(k)-E(k)\Big\}-K(k)\right], \tag{9}$$

where

$$k^2 = \frac{4a\rho}{z^2+(\rho+a)^2}. \tag{10}$$

The magnetic field is

$$\begin{aligned}\mathbf{B}(\mathbf{r}) = {} & \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\left[K(k)-\frac{(z^2+\rho^2-a^2)}{z^2+(\rho-a)^2}E(k)\right] \\ & -\hat{\boldsymbol{\rho}}\frac{\mu_0 I}{4\pi}\frac{2}{\sqrt{z^2+(\rho+a)^2}}\frac{z}{\rho}\left[K(k)-\frac{(z^2+\rho^2+a^2)}{z^2+(\rho-a)^2}E(k)\right].\end{aligned} \tag{11}$$

Evaluate the vector potential and the magnetic field on the symmetry axis of the loop, obtained by setting $\rho \to 0$ in the above expressions.
Hint: For the $\hat{\boldsymbol{\rho}}$ direction keep terms to order k^4 for elliptic functions. Further, observe that

$$\frac{(z^2+\rho^2+a^2)}{z^2+(\rho-a)^2} = \frac{(2-k^2)}{2(1-k^2)}. \tag{12}$$