Homework No. 09 (2021 Spring)

PHYS 420: ELECTRICITY AND MAGNETISM II

Department of Physics, Southern Illinois University–Carbondale

Due date: Monday, 2021 Apr 12, 2:00 PM

0. (**0 points.**) Keywords for finding resource materials: Electrodynamics of moving bodies; Retarded time; Delta functions.

0. Problems 1 and 2 are to be submitted for assessment. Rest are for practice.

1. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{1}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$, evaluate

$$\delta(ax^2+bx+c). \tag{2}$$

2. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{3}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$, evaluate

$$\delta(x^3-6x^2+11x-6). \tag{4}$$

3. (**20 points.**) The electric and magnetic field generated by a particle with charge q moving along the z axis with speed v, $\beta = v/c$, can be expressed in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\left[x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z-vt)\hat{\mathbf{k}}\right]}{(x^2+y^2)}\frac{(x^2+y^2)(1-\beta^2)}{\left[(x^2+y^2)(1-\beta^2)+(z-vt)^2\right]^{\frac{3}{2}}}, \tag{5a}$$

$$c\mathbf{B}(\mathbf{r},t) = \boldsymbol{\beta}\times\mathbf{E}(\mathbf{r},t). \tag{5b}$$

   (a) Consider the distribution

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{2}\frac{\epsilon}{(x^2+\epsilon)^{\frac{3}{2}}}. \tag{6}$$

   Show that

$$\delta(x)\begin{cases}\to \frac{1}{2\sqrt{\epsilon}} \to \infty, & \text{if} \quad x=0,\\ \to \frac{\epsilon}{2x^3} \to 0, & \text{if} \quad x\neq 0.\end{cases} \tag{7}$$

   Further, show that

$$\int_{-\infty}^{\infty} dx\,\delta(x) = 1. \tag{8}$$

(b) Thus, verify that the electric and magnetic field of a charge approaching the speed of light can be expressed in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}\,\delta(z-ct), \tag{9a}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct) = 2cq\frac{\mu_0}{4\pi}\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct), \tag{9b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\rho = \sqrt{x^2+y^2}$. These fields are confined on the $z = ct$ plane moving with speed c. Illustrate this field configuration using a diagram.