Final Exam (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

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1. (**20 points.**) Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \tag{1}$$

and the corresponding Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \tag{2}$$

For the case when the total energy E is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \qquad r_0 = \frac{L_z^2}{\mu\alpha}, \tag{3}$$

where L_z is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \qquad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \tag{4}$$

Perihelion is the point in the orbit of a planet when it is closest to the Sun. This corresponds to $\phi = \phi_0$. The precession of the perihelion is suitably defined in terms of the angular displacement $\Delta\phi$ of the perihelion during one revolution,

$$\Delta\phi = 2\left[\int_{r_{\text{min}}}^{r_{\text{max}}} d\phi\right] - 2\pi, \tag{5}$$

where

$$r_{\text{min}} = \frac{r_0}{1+e} \tag{6}$$

is the perihelion, when the planet is closest to Sun, and

$$r_{\text{max}} = \frac{r_0}{1-e} \tag{7}$$

is the aphelion, corresponding to $\phi = \phi_0 + \pi$, when the planet is farthest from Sun. For the Kepler problem derive the relation

$$d\phi = \frac{r_0\,dr}{r^2}\frac{1}{\sqrt{e^2 - \left(1 - \frac{r_0}{r}\right)^2}}. \tag{8}$$

Show that the precession of perihelion is zero for the Kepler problem.

2. (**20 points.**) Time dilates. That is,

$$T = T_0 \gamma, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{9}$$

where T_0 is the proper time measured in the instantaneous rest frame of the clock measuring T_0 and T is the time measured by a clock moving with velocity v relative to the clock measuring proper time. Similarly, show that (for $\mathbf{v} \,||\, \mathbf{a}$)

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{10}$$

where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneous rest frame of the particle. Derive the equation for the trajectory of a particle moving in a straight line (along the z axis) with constant proper acceleration, after starting from rest from the point $z = c^2/|\mathbf{a}_0|$ at time $t = 0$.