Homework No. 01 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Jan 26, 9.30am

1. (**20 points.**) A simple pendulum consists of a particle of mass m suspended by a massless rod of length l in a uniform gravitational field g.

   (a) Identify the two forces acting on the pendulum to be the force of gravity $m\mathbf{g}$ and the force of tension $\mathbf{T}$. Thus, deduce the Newton equation of motion to be

   $$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \tag{1}$$

   where $\mathbf{a}$ is acceleration of mass m. Starting from Eq. (1) derive the equation of motion for the simple pendulum

   $$\frac{d^2\phi}{dt^2} = -\omega_0^2 \sin\phi, \tag{2}$$

   where

   $$\omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{g}{l}}. \tag{3}$$

   (b) Starting from Eq. (2) derive the statement of conservation of energy for this system,

   $$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = \text{constant}. \tag{4}$$

   Hint: Multiply Eq. (2) by $\dot{\phi}$ and express the equation as a total derivative with respect to time.

   (c) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that

   $$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0. \tag{5}$$

   Thus, derive

   $$\frac{dt}{T_0} = \frac{1}{2\pi}\frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{6}$$

   where $T_0 = 2\pi\sqrt{l/g}$.

(d) The time period of oscillations of the simple pendulum is equal to four times the time taken between $\phi = 0$ and $\phi = \phi_0$. Thus, show that

$$T = 4\frac{T_0}{2\pi}\int_0^{\phi_0} \frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{7}$$

$$= \frac{T_0}{\pi}\int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}}. \tag{8}$$

Then, substitute $\sin\theta = \sin(\phi/2)/\sin(\phi_0/2)$ to determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0\,\frac{2}{\pi}K\left(\sin\frac{\phi_0}{2}\right), \tag{9}$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{10}$$

is the complete elliptic integral of the first kind.

(e) Using the power series expansion

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} \tag{11}$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0\left[1 + \frac{\phi_0^2}{16} + \dots\right]. \tag{12}$$

(f) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60°$.

(g) Plot the time period T of Eq. (9) as a function of ϕ_0. What can you conclude about the time period for $\phi_0 = \pi$?