Homework No. 04 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Feb 16, 9.30am

1. (**20 points.**) Fermat's principle in ray optics states that a ray of light takes the path of least time between two given points. The speed of light in a medium is given in terms of the refractive index

$$n = \frac{c}{v}, \tag{1}$$

of the medium, where c is the speed of light in vacuum and v is the speed of light in the medium. Consider a ray of light traversing a path from (x_1, y_1) to (x_2, y_2) in a plane of fixed z.

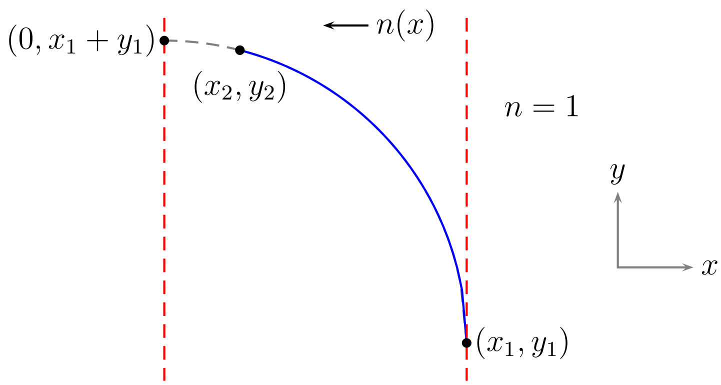

Figure 1: Problem 1.

(a) Show that the time taken to travel an infinitesimal distance ds is given by

$$dt = \frac{ds}{v} = \frac{n\,ds}{c}, \tag{2}$$

where ds in a plane is characterized by the infinitesimal statement

$$ds^2 = dx^2 + dy^2. \tag{3}$$

(b) Fermat's principle states that the path traversed by a ray of light from (x_1, y_1) to (x_2, y_2) is the extremal of the functional

$$T[y] = \frac{1}{c}\int_{(x_1,y_1)}^{(x_2,y_2)} n\,ds = \frac{1}{c}\int_{x_1}^{x_2} dx\, n(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4}$$

(c) Since the ray of light passes through the points (x_1, y_1) and (x_2, y_2), we do not consider variations at these (end) points. Thus, show that

$$\frac{\delta T[y]}{\delta y(x)} = -\frac{d}{dx}\left[\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\right]. \tag{5}$$

(d) Using Fermat's principle show that the differential equation for the path $y(x)$ traversed by the ray of light is

$$\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = n_0, \tag{6}$$

where n_0 is a constant. Show that the above equation can be rewritten in the form

$$\frac{dy}{dx} = \frac{n_0}{\sqrt{n(x)^2 - n_0^2}}. \tag{7}$$

(e) Let us consider a medium with refractive index $(x_1 = a)$

$$n(x) = \begin{cases} \dfrac{a}{x}, & 0 < x < a, \\ 1, & a < x. \end{cases} \tag{8}$$

Solve the corresponding differential equation to obtain

$$y(x) - y_0 = \frac{1}{n_0}\left[\sqrt{a^2 - n_0^2 x^2} - \sqrt{a^2 - n_0^2 a^2}\right], \qquad x < a. \tag{9}$$

The path in this medium satisfies the equation of a circle. Determine the radius of the circle to be a/n_0 and the location of the center to be $(0, y_0 - a\sqrt{(1/n_0^2) - 1})$. For initial conditions

$$y(x_1) = y_1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} = y_1' \tag{10}$$

show that the integration constants are determined to be

$$y_0 = y_1 \qquad \text{and} \qquad n_0 = \frac{y_1'}{\sqrt{1 + y_1'^2}}. \tag{11}$$

For the special case when $y_1 = 0$ and $y_1' \to \infty$ show that $n_0 = 1$ and

$$y(x) = \sqrt{a^2 - x^2}, \qquad x < a. \tag{12}$$

Evaluate the total time taken for light to go from $(x_1 = a, y_1 = 0)$ to $(x_2 = 0, y_2 = a)$.