Homework No. 05 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Mar 2, 9.30am

1. (**60 points.**) Consider a rope of uniform mass density $\lambda = dm/ds$ hanging from two points, (x_1, y_1) and (x_2, y_2), as shown in Figure 1. The gravitational potential energy of

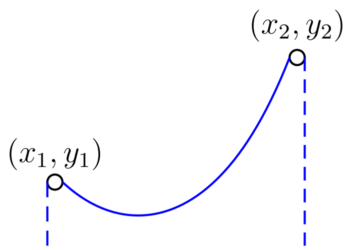

Figure 1: Problem 1.

an infinitely tiny element of this rope at point (x, y) is given by

$$dU = dm\, gy = \lambda g ds\, y, \tag{1}$$

where

$$ds^2 = dx^2 + dy^2. \tag{2}$$

A catenary is the curve that the rope assumes, that minimizes the total potential energy of the rope.

(a) Show that the total potential energy U of the rope hanging between points x_1 and x_2 is given by

$$U[x] = \lambda g \int_{(x_1,y_1)}^{(x_2,y_2)} y ds = \lambda g \int_{y_1}^{y_2} dy\, y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{3}$$

(b) Since the curve passes through the points (x_1, y_1) and (x_2, y_2), we have no variations at these (end) points. Thus, show that

$$\frac{\delta U[x]}{\delta x(y)} = -\lambda g \frac{d}{dy} \left[y \frac{\frac{dx}{dy}}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} \right]. \tag{4}$$

(c) Using the extremum principle show that the differential equation for the catenary is

$$\frac{dx}{dy} = \frac{a}{\sqrt{y^2 - a^2}}, \tag{5}$$

where a is an integration contant.

(d) Show that integration of the differential equation yields the equation of the catenary

$$y = a \cosh \frac{x - x_0}{a}, \tag{6}$$

where x_0 is another integration constant.

(e) For the case $y_1 = y_2$ we have

$$\frac{y_1}{a} = \cosh \frac{x_1 - x_0}{a}, \tag{7a}$$

$$\frac{y_2}{a} = \cosh \frac{x_2 - x_0}{a}, \tag{7b}$$

which leads to the solution, assuming $x_1 \neq x_2$,

$$x_0 = \frac{x_1 + x_2}{2}. \tag{8}$$

Identify x_0 in Figure 1.

(f) Next, derive

$$\frac{y_1}{a} = \frac{y_2}{a} = \cosh \frac{x_2 - x_1}{2a}, \tag{9}$$

which, in principle, determines a. However, this is a transcendental equation in a and does not allow exact evaluation of a, and one depends on numerical solutions. Observe that, if $x = x_0$ in Eq. (6), then $y = a$. Identify a in Figure 1.

2. (**20 points.**) A catenary is described by

$$y = a \cosh \left(\frac{x - x_0}{a} \right), \tag{10}$$

where constants a and x_0 are determined by the position of the end points (x_1, y_1) and (x_2, y_2). Let us choose $x_0 = 0$ and $a = 1$ such that

$$y = \cosh x, \tag{11}$$

where x and y are dimensionless variables.

(a) Using series expansion show that

$$\cosh x = 1 + \frac{x^2}{2} + \ldots. \tag{12}$$

(b) The parabola

$$y = 1 + \frac{x^2}{2} \tag{13}$$

is an approximation for the catenary. Plot the above parabola and a catenary in the same plot for $-1 < x < 1$ and estimate the maximum error in the approximation.