

Homework No. 06 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2021 Mar 11, 9.30am

1. **(20 points.)** (Refer Goldstein, 2nd edition, Chapter 1 Problem 8.) As a consequence of the Hamilton's stationary action principle, the equations of motion for a system can be expressed as Euler-Lagrange equations,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \quad (1)$$

in terms of a Lagrangian $L(x, \dot{x}, t)$. Show that the Lagrangian for a system is not unique. In particular, show that if $L(x, \dot{x}, t)$ satisfies the Euler-Lagrange equation then

$$L'(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{dF(x, t)}{dt}, \quad (2)$$

where $F(x, t)$ is any arbitrary differentiable function, also satisfies the Euler-Lagrange equation.

2. **(20 points.)** A mass m_1 is forced to move on a vertical circle of radius R with uniform angular speed ω . Another mass m_2 is connected to mass m_1 using a massless rod of length a , such that it is a simple pendulum with respect to mass m_1 . Motion of both the masses is constrained to be in a vertical plane in a uniform gravitational field.

- (a) Write the Lagrangian for the system.
- (b) Determine the equation of motion for the system.
- (c) Give physical interpretation of each term in the equation of motion.

3. **(20 points.)** A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a . The pivot, in general, has mass m_1 , but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 1. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

- (a) Determine the Lagrangian for the system to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2a^2\dot{\theta}^2 + m_2a\dot{x}\dot{\theta} \cos \theta + m_2ga \cos \theta. \quad (3)$$

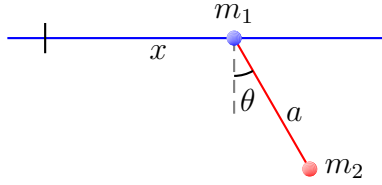


Figure 1: Problem 3.

- (b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{x}} = m_2 \dot{x} + m_2 a \dot{\theta} \cos \theta, \quad (4a)$$

$$\frac{\partial L}{\partial x} = 0, \quad (4b)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 a^2 \dot{\theta} + m_2 a \dot{x} \cos \theta, \quad (4c)$$

$$\frac{\partial L}{\partial \theta} = -m_2 a \dot{x} \dot{\theta} \sin \theta - m_2 g a \sin \theta. \quad (4d)$$

- (c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a \ddot{\theta} \cos \theta - a \dot{\theta}^2 \sin \theta = 0, \quad (5a)$$

$$a \ddot{\theta} + \ddot{x} \cos \theta + g \sin \theta = 0. \quad (5b)$$

Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

- (d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{x} + a \ddot{\theta} = 0, \quad (6a)$$

$$a \ddot{\theta} + \ddot{x} + g \theta = 0. \quad (6b)$$

Determine the solution to be given by

$$\theta = 0 \quad \text{and} \quad \ddot{x} = 0. \quad (7)$$

Interpret this solution.

- (e) The solution $\theta = 0$ seems to be too restrictive. Will this system not allow $\theta \neq 0$? To investigate this, let us not restrict to the small angle approximation. Rewrite Eqs. (5), using Eq. (5a) in Eq. (5b), as

$$\ddot{x} + a \ddot{\theta} \cos \theta - a \dot{\theta}^2 \sin \theta = 0, \quad (8a)$$

$$\sin \theta \left[a \ddot{\theta} \sin \theta + a \dot{\theta}^2 \cos \theta + g \right] = 0. \quad (8b)$$

In this form we immediately observe that $\theta = 0$ is a solution. However, it is not the only solution. Towards interpreting Eqs. (8) let us identify the coordinates of the center of mass of the m_1 - m_2 system,

$$(m_1 + m_2)x_{\text{cm}} = m_1x + m_2(x + a \sin \theta), \quad (9a)$$

$$(m_1 + m_2)y_{\text{cm}} = -m_2a \cos \theta, \quad (9b)$$

which for $m_1 = 0$ are the coordinates of the mass m_2 ,

$$x_{\text{cm}} = x + a \sin \theta, \quad (10a)$$

$$y_{\text{cm}} = -a \cos \theta. \quad (10b)$$

Show that

$$\dot{x}_{\text{cm}} = \dot{x} + a\dot{\theta} \cos \theta, \quad (11a)$$

$$\dot{y}_{\text{cm}} = a\dot{\theta} \sin \theta, \quad (11b)$$

and

$$\ddot{x}_{\text{cm}} = \ddot{x} + a\ddot{\theta} \cos \theta - a\dot{\theta}^2 \sin \theta, \quad (12a)$$

$$\ddot{y}_{\text{cm}} = a\ddot{\theta} \sin \theta + a\dot{\theta}^2 \cos \theta. \quad (12b)$$

Comparing Eqs. (8) and Eqs. (12) we learn that

$$\ddot{x}_{\text{cm}} = 0, \quad (13a)$$

$$\sin \theta [\ddot{y}_{\text{cm}} + g] = 0. \quad (13b)$$

Thus, $\ddot{y}_{\text{cm}} = -g$ is the more general solution, and $\theta = 0$ is a trivial solution.

- (f) Let us analyse the system for initial conditions: $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$, $\dot{x}(0) = 0$. Show that for this case $\dot{x}_{\text{cm}}(0) = 0$ and

$$a(\cos \theta - \cos \theta_0) = \frac{1}{2}gt^2. \quad (14)$$

Plot θ as a function of time t . Interpret this solution.