# Homework No. 07 (2021 Spring)

## PHYS 510: CLASSICAL MECHANICS

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Due date: Tuesday, 2021 Mar 22, 9.30am

1. (**20 points.**) A pendulum consists of a mass $m_2$ hanging from a pivot by a massless string of length $a_2$. The pivot, in general, has mass $m_1$, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a frictionless hoop of radius $a_1$. See Figure 1. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.

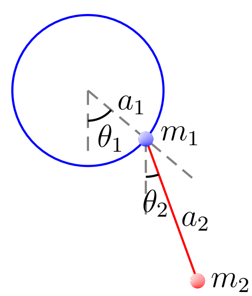


Figure 1: Problem 1.

(a) Determine the Lagrangian for the system to be

$$L(\theta_1, \dot\theta_1, \theta_2, \dot\theta_2) = \frac{1}{2}m_2 a_1^2\dot\theta_1^2 + \frac{1}{2}m_2 a_2^2\dot\theta_2^2 + m_2 a_1 a_2 \dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2) + m_2 g a_1 \cos\theta_1 + m_2 g a_2 \cos\theta_2. \tag{1}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot\theta_1} = m_2 a_1^2 \dot\theta_1 + m_2 a_1 a_2 \dot\theta_2 \cos(\theta_1 - \theta_2), \tag{2a}$$
$$\frac{\partial L}{\partial \theta_1} = -m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g a_1 \sin\theta_1, \tag{2b}$$
$$\frac{\partial L}{\partial \dot\theta_2} = m_2 a_2^2 \dot\theta_2 + m_2 a_1 a_2 \dot\theta_1 \cos(\theta_1 - \theta_2), \tag{2c}$$
$$\frac{\partial L}{\partial \theta_2} = m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g a_2 \sin\theta_2. \tag{2d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{\theta}_1 + \omega_1^2 \sin\theta_1 + \frac{1}{\beta}\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta}\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{3a}$$

$$\ddot{\theta}_2 + \omega_2^2 \sin\theta_2 + \beta\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \beta\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0, \tag{3b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{4}$$

Note that $\beta$ is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass $m_2$ when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \frac{1}{\beta}\ddot{\theta}_2 = 0, \tag{5a}$$

$$\ddot{\theta}_2 + \omega_2^2\theta_2 + \beta\ddot{\theta}_1 = 0. \tag{5b}$$

(e) Determine the solution for the initial conditions

$$\theta_1(0) = \theta_2(0) = \theta_{20}, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0. \tag{6}$$

Interpret and expound your solution.

2. (**20 points.**) Consider the coplanar double pendulum in Figure 2.

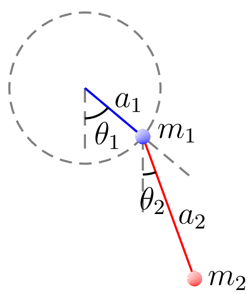


Figure 2: Problem 2.

(a) Write the Lagrangian for the system. in particular, show that the Lagrangian can be expressed in the form

$$L = L_1 + L_2 + L_{\text{int}}, \tag{7}$$

where

$$L_1 = \frac{1}{2}(m_1 + m_2)a_1^2\dot{\theta}_1^2 + (m_1 + m_2)ga_1\cos\theta_1, \tag{8a}$$

$$L_2 = \frac{1}{2}m_2 a_2^2\dot{\theta}_2^2 + m_2 g a_2 \cos\theta_2, \tag{8b}$$

$$L_{\text{int}} = m_2 a_1 a_2 \dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2). \tag{8c}$$

(b) Determine the equations of motion for the system. Express them in the form

$$(m_1+m_2)a_1\ddot{\theta}_1+(m_1+m_2)g\sin\theta_1+m_2a_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+m_2a_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)=0, \tag{9a}$$
$$a_2\ddot{\theta}_2+g\sin\theta_2+a_1\ddot{\theta}_1\cos(\theta_1-\theta_2)-a_1\dot{\theta}_1^2\sin(\theta_1-\theta_2)=0. \tag{9b}$$

(c) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1+\omega_1^2\theta_1+\frac{\alpha}{\beta}\ddot{\theta}_2=0, \tag{10a}$$
$$\ddot{\theta}_2+\omega_2^2\theta_1+\beta\ddot{\theta}_1=0, \tag{10b}$$

where

$$\omega_1^2=\frac{g}{a_1},\quad \omega_2^2=\frac{g}{a_2},\quad \alpha=\frac{m_2}{m_1+m_2},\quad \beta=\frac{a_1}{a_2}=\frac{\omega_2^2}{\omega_1^2}. \tag{11}$$

Note that $0\le\alpha\le 1$.

(d) Determine the solution for the initial conditions

$$\theta_1(0)=0,\quad \theta_2(0)=0,\quad \dot{\theta}_1(0)=0,\quad \dot{\theta}_2(0)=\omega_0, \tag{12}$$

for $\alpha=1/2$ and $\beta=1$.