Homework No. 09 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Apr 6, 9.30am

1. (**20 points.**) Consider infinitesimal rigid translation in space, described by

$$\delta \mathbf{r} = \delta \boldsymbol{\epsilon}, \quad \delta \mathbf{p} = 0, \quad \delta t = 0, \tag{1}$$

where $\delta \boldsymbol{\epsilon}$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial \mathbf{r}}. \tag{2}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{p}}{dt} = \mathbf{p}(t_2) - \mathbf{p}(t_1). \tag{3}$$

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial \mathbf{r}} = 0. \tag{4}$$

That is, when the Hamiltonian is independent of position. Or, when the force $\mathbf{F} = -\partial H / \partial \mathbf{r} = 0$.

(d) Deduce that the linear momentum is conserved, that is,

$$\mathbf{p}(t_1) = \mathbf{p}(t_2), \tag{5}$$

when the action has translation symmetry.

2. (**20 points.**) Consider infinitesimal rigid translation in time, described by

$$\delta \mathbf{r} = 0, \quad \delta \mathbf{p} = 0, \quad \delta t = \delta \epsilon, \tag{6}$$

where $\delta \epsilon$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \epsilon} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial t}. \tag{7}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \epsilon} = \int_{t_1}^{t_2} dt \frac{dH}{dt} = H(t_2) - H(t_1). \tag{8}$$

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial t} = 0. \tag{9}$$

That is, when the Hamiltonian is independent of time.

(d) Deduce that the Hamiltonian is conserved, that is,

$$H(t_1) = H(t_2), \tag{10}$$

when the action has translation symmetry.

3. (**20 points.**) Consider infinitesimal rigid rotation, described by

$$\delta \mathbf{r} = \delta \boldsymbol{\omega} \times \mathbf{r}, \quad \delta \mathbf{p} = \delta \boldsymbol{\omega} \times \mathbf{p}, \quad \delta t = 0, \tag{11}$$

where $d\delta\boldsymbol{\omega}/dt = 0$.

(a) Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta \boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} \right] \tag{12}$$

or

$$\frac{\delta W}{\delta \boldsymbol{\omega}} = -\int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} \right]. \tag{13}$$

(b) Show, separately, that the change in the action under the above rotation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\omega}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{L}}{dt} = \mathbf{L}(t_2) - \mathbf{L}(t_1), \tag{14}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum.

(c) The system is defined to have rotational symmetry when the action does not change under rigid rotation. Show that a system has rotation symmetry when

$$\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} = 0, \tag{15}$$

or

$$\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} = 0. \tag{16}$$

Show that this corresponds to

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \phi} = 0, \tag{17}$$

or

$$\frac{\partial H}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \phi} = 0. \tag{18}$$

That is, when the Lagrangian is independent of angular coordinates θ and ϕ.

(d) Deduce that the anglular momentum is conserved, that is,

$$\mathbf{L}(t_1) = \mathbf{L}(t_2), \tag{19}$$

when the action has rotational symmetry.