Homework No. 10 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Apr 6, 9.30am

1. (**20 points.**) Spherical pendulum: Consider a pendulum that is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The mass is then subject to the condition of constraint

$$F = \frac{1}{2}(x^2 + y^2 + z^2 - a^2) = 0, \tag{1}$$

where the factor of 1/2 is introduced anticipating cancellations. Consider the Lagrangian function

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - mgz - \lambda F. \tag{2}$$

(a) Evaluate the gradient $\boldsymbol{\nabla}$ of the condition of constraint. Show that

$$\boldsymbol{\nabla} F = \mathbf{r}. \tag{3}$$

(b) Using the Euler-Lagrange equations derive the equations of motion

$$m\ddot{\mathbf{r}} = -mg\hat{\mathbf{z}} - \lambda\mathbf{r}. \tag{4}$$

(c) Derive an expression for λ. In particular, show that it can be expressed in the form

$$-\lambda a = \hat{\mathbf{r}} \cdot \mathbf{N}. \tag{5}$$

Find $\mathbf{N}$. Give the physical interpretation of $\mathbf{N}$ using D'Alembert's principle.

(d) Show that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum of the particle, about the z-axis is conserved. That is,

$$\frac{d}{dt}(\hat{\mathbf{z}} \cdot \mathbf{L}) = 0. \tag{6}$$

Show that this also implies the conservation of the areal velocity

$$\frac{dS}{dt} = \frac{1}{2}(x\dot{y} - y\dot{x}), \tag{7}$$

where S is the area swept out.

(e) Show that

$$\frac{dF}{dt} = \mathbf{r} \cdot \dot{\mathbf{r}} = 0. \tag{8}$$

Using this derive the statement of conservation of energy,

$$\frac{dH}{dt} = 0, \qquad H = \frac{1}{2} m \dot{\mathbf{r}}^2 + mgz, \tag{9}$$

starting from the equation of motion in Eq. (4) and multiplying by $\dot{\mathbf{r}}$.

2. (**20 points.**) Consider a wheel rolling on a horizontal surface. The following distinct

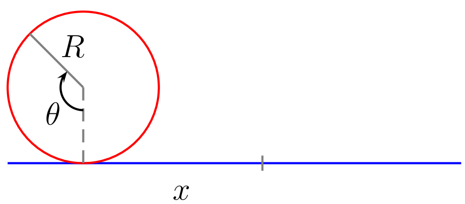

Figure 1: Problem 2.

types of motion are possible for the wheel:

$$\begin{aligned} x &< \theta R, && \text{slipping (e.g. in snow)}, \\ x &= \theta R, && \text{perfect rolling}, \\ x &> \theta R, && \text{sliding (e.g. on ice)}. \end{aligned} \tag{10}$$

Differentiation of the these relations leads to the characterizations, $v < \omega R$, $v = \omega R$, and $v > \omega R$, respectively, where $v = \dot{x}$ is the linear velocity and $\omega = \dot{\theta}$ is the angular velocity. Assuming the wheel is perfectly rolling, at a given instant of time, the tendency of motion could be to slip, to keep on perfectly rolling, or to slide.

Deduce that while perfectly rolling the relative motion of the point on the wheel that is in contact with the surface with respect to the surface is exactly zero. Thus, conclude that the force of friction on the wheel is zero. The analogy is a mass at rest on a horizontal surface. However, while perfectly rolling, it is possible to have the tendency to slip or slide without actually slipping of sliding. The analogy is that of a mass at rest under the action of an external force and the force of friction. In these cases the force of friction is that of static friction and it acts in the forward or backward direction.

In the following we differentiate between the following:

(a) Tendency of the wheel is to slip (without actually slipping) while perfectly rolling.

(b) Tendency of the wheel is to keep on perfectly rolling.

(c) Tendency of the wheel is to slide (without actually sliding) while perfectly rolling.

Deduce the direction of the force of friction in the above cases. Determine if the friction is working against linear acceleration or angular acceleration.

Perfect rolling involves the contraint $x = \theta R$. Thus, using the D'Alembert principle and the idea of Lagrange multiplier we can write the Lagragian for a perfectly rolling wheel on a horizontal surface to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 - F_s (x - \theta R), \tag{11}$$

where m is the mass of the wheel, I is the moment of inertia of the wheel, and F_s is the Lagrangian multiplier. Using D'Alembert's principle give an interpretation for the Lagrangian multiplier F_s. What is the dimension of F_s? Infer the sign of F_s for the cases when the tendency of the wheel is to slip or slide while perfectly rolling.