Homework No. 12 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2021 Apr 22, 9.30am

1. (**20 points.**) Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \tag{1}$$

and the corresponding Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \tag{2}$$

For the case when the total energy E is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \qquad r_0 = \frac{L_z^2}{\mu\alpha}, \tag{3}$$

where L_z is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \qquad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \tag{4}$$

Perihelion is the point in the orbit of a planet when it is closest to the Sun. This corresponds to $\phi = \phi_0$. The precession of the perihelion is suitably defined in terms of the angular displacement $\Delta\phi$ of the perihelion during one revolution,

$$\Delta\phi = 2\left[\int_{r_{\min}}^{r_{\max}} d\phi\right] - 2\pi, \tag{5}$$

where

$$r_{\min} = \frac{r_0}{1+e} \tag{6}$$

is the perihelion, when the planet is closest to Sun, and

$$r_{\max} = \frac{r_0}{1-e} \tag{7}$$

is the aphelion, corresponding to $\phi = \phi_0 + \pi$, when the planet is farthest from Sun.

(a) For the Kepler problem derive the relation

$$d\phi = \frac{r_0\, dr}{r^2} \frac{1}{\sqrt{e^2 - \left(1 - \frac{r_0}{r}\right)^2}}. \tag{8}$$

Show that the precession of perihelion is zero for the Kepler problem.

(b) When a small correction

$$\delta U(r) = -\frac{\beta}{r^3} = \kappa\, U_0 \left(\frac{r_0}{r}\right)^3, \tag{9}$$

expressed in terms of dimensionless parameter κ using the relation $\beta = -\kappa U_0 r_0^3$, is added we have the perturbed potential energy

$$U(r) = -\frac{\alpha}{r} - \frac{\beta}{r^3} = -\frac{\alpha}{2r_0}\left[\frac{r_0}{r} + \kappa \left(\frac{r_0}{r}\right)^3\right]. \tag{10}$$

Show that the precession of the perihelion is no longer zero. Repeat the steps in the lecture dated 2021 April 13 and calculate the precession of the perihelion for this case.