Homework No. 13 (2021 Spring)

PHYS 510: CLASSICAL MECHANICS

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2021 Apr 29, 9.30am

1. (**20 points.**) (Refer Hughston and Tod's book.) Prove that

   (a) if p_μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.

   (b) if p_μ and q^μ are both time-like vectors and $p^\mu q_\mu < 0$ then either both are future-pointing or both are past-pointing.

2. (**20 points.**) Let

$$\tanh\theta = \beta, \tag{1}$$

   where $\beta = v/c$. Addition of (parallel) velocities in terms of the parameter θ obeys the arithmatic addition

$$\theta = \theta_a + \theta_b. \tag{2}$$

   (a) Invert the expression in Eq. (1) to find the explicit form of θ in terms of β as a logarithm.

   (b) Show that Eq. (2) leads to the relation

$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta_a}{1-\beta_a}\right)\left(\frac{1+\beta_b}{1-\beta_b}\right). \tag{3}$$

   (c) Using Eq. (3) derive the Poincaré formula for the addition of (parallel) velocities.

3. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{4}$$

   (a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 3. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{5}$$

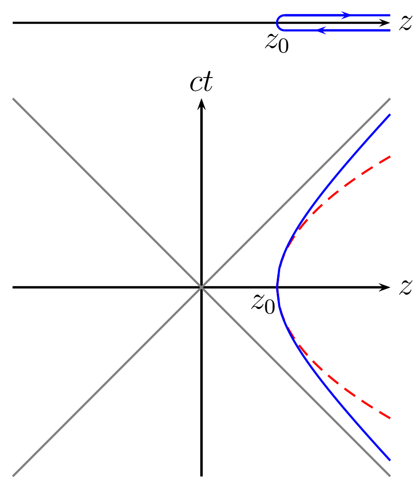

Figure 1: Problem 3

as described by the dashed (red) curve in the space-time diagram in Figure 3. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2t^2}{z_0^2}} \tag{6}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{7}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{8}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. Plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.

4. (**20 points.**) The path of a relativistic particle 1 moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_1(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{9}$$

This is the motion of a particle that comes to existance at $z_1 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Consider another relavistic particle 2 undergoing hyperbolic motion given by

$$z_2(t) = -\sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{10}$$

This is the motion of a particle that comes to existance at $z_2 = -\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = -z_0$, and then returns back to negative infinity. The world-line of particle 1 is the blue curve in Figure 4, and the world-line of particle 2 is the red curve in Figure 4. Using geometric (diagrammatic) arguments might be easiest to answer the following. Imagine the particles are sources of light (imagine a flash light pointing towards origin).

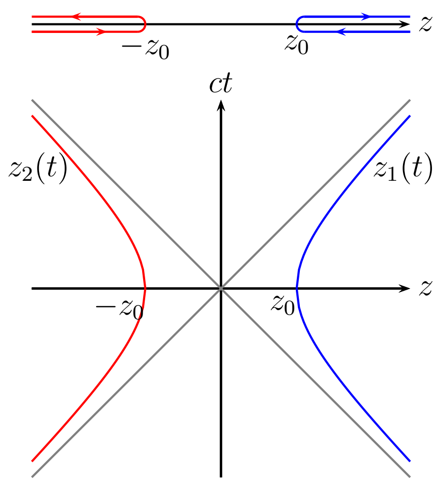

Figure 2: Problem 4

(a) At what time will the light from particle 1 first reach particle 2? Where are the particles when this happens?

(b) At what time will the light from particle 2 first reach particle 1? Where are the particles when this happens?

(c) Can the particles communicate with each other?

(d) Can the particles ever detect the presence of the other? In other words, can one particle be aware of the existence of the other? What can you deduce about the observable part of our universe from this analysis?