Midterm Exam No. 01 (2021 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Feb 23, 12.30pm

1. (**20 points.**) Determine the magnetic field due to a point charge moving with uniform velocity.

2. (**20 points.**) Using Biot-Savart law determine the magnetic field due to an infinitely long wire carrying a steady current.

3. (**20 points.**) It is a bit perplexing that the magnetic field due to an infinitely long solenoid is independent of the radius of the solenoid. It prompts us to investigate the limiting case when the radius of the solenoid goes to zero. Such a solenoid can be imagined to be built out of point magnetic moments stacked up along the z axis with all their moments pointing along the z axis. Let us define the magnetic moment per unit length for a such an infinitely thin 'solenoid' to be

$$\frac{\text{magnetic moment}}{\text{length}} = \boldsymbol{\lambda} = \hat{\mathbf{z}}\frac{d\mathbf{m}}{dz}. \tag{1}$$

A Dirac string constitutes of an infinitely thin solenoid that extends from a point $\mathbf{r}'$ to infinity along an arbitrary curve. Let us consider a Dirac string that extends from the origin to infinity along a straight line, the negative z axis. Thus, we can write the magnetic moment density

$$\hat{\mathbf{z}}\,\lambda\delta(x)\delta(y). \tag{2}$$

(a) The magnetic vector potential for an infinitesimal element of the 'solenoid' is that of a point magnetic dipole given by

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\frac{d\mathbf{m}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\frac{\lambda\rho\,dz'}{[\rho^2+(z-z')^2]^{\frac{3}{2}}}. \tag{3}$$

Thus, the total magnetic vector potential at any point is obtained by the vector sum of all the elements, using integration,

$$\mathbf{A}(\mathbf{r}) = \int d\mathbf{A} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\int_{-\infty}^{0}\frac{\lambda\rho\,dz'}{[\rho^2+(z-z')^2]^{\frac{3}{2}}}. \tag{4}$$

Complete the integral and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\frac{\lambda}{r}\frac{(1-\cos\theta)}{\sin\theta}. \tag{5}$$

(b) Evaluate the magnetic field for this magnetic vector potential using the relation

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{6}$$

and show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\lambda\frac{\hat{\mathbf{r}}}{r^2}. \tag{7}$$

Compare this to the magnetic field due to a magnetic monopole.

(c) Evaluate

$$\boldsymbol{\nabla} \cdot \mathbf{B}. \tag{8}$$

Is it zero?

4. (**20 points.**) Verify that

$$K(0) = \frac{\pi}{2}, \tag{9a}$$

$$E(0) = \frac{\pi}{2}. \tag{9b}$$

Then, verify that

$$E(1) = 1. \tag{10}$$

Note that

$$K(1) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\cos\psi} \tag{11}$$

is divergent. To see the nature of divergence we introduce a cutoff parameter $\delta > 0$ and write

$$K(1) = \int_0^{\frac{\pi}{2}-\delta} \frac{d\psi}{\cos\psi}. \tag{12}$$

Evaluate the integral, (using the identity $d(\sec\psi + \tan\psi)/d\psi = \sec\psi(\sec\psi + \tan\psi)$,) and show that

$$K(1) \sim \ln 2 - \ln\delta - \frac{\delta^2}{12} + \mathcal{O}(\delta)^4 \tag{13}$$

has logarithmic divergence. Using Mathematica (or another graphing tool) plot $K(k)$ and $E(k)$ as functions of k for $0 \leq k < 1$.