Homework No. 01 (2021 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Jan 26, 12.30pm

1. (**20 points.**) The solution to the Maxwell equations for the case of magnetostatics in terms of the vector potential $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1}$$

   (a) Verify that the above solution satisfies the Coulomb gauge condition. That is, it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{2}$$

   (b) Further, verify that the magnetic field is the curl of the vector potential and can be expressed in the form

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{3}$$

2. (**20 points.**) The solution to the Maxwell equations for the case of magnetostatics was found to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{4}$$

   Verify that the above solution satisfies magnetostatics equations, that is, it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{5}$$

   and

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{6}$$

3. (**50 points.**) (Based on Problem 5.8, Griffiths 4th edition.)
   The magnetic field at position $\mathbf{r} = (x, y, z)$ due to a finite wire segment of length $2L$ carrying a steady current I, with the caveat that it is unrealistic (why?), placed on the z-axis with its end points at $(0, 0, L)$ and $(0, 0, -L)$, is

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} \frac{1}{\sqrt{x^2+y^2}} \left[\frac{z+L}{\sqrt{x^2+y^2+(z+L)^2}} - \frac{z-L}{\sqrt{x^2+y^2+(z-L)^2}} \right], \tag{7}$$

   where $\hat{\boldsymbol{\phi}} = (-\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}) = (-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}})/\sqrt{x^2+y^2}$.

(a) Show that by taking the limit $L \to \infty$ we obtain the magnetic field near a long straight wire carrying a steady current I,

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}, \tag{8}$$

where $\rho = \sqrt{x^2 + y^2}$ is the perpendicular distance from the wire.

(b) Show that the magnetic field on a line bisecting the wire segment is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}\frac{L}{\sqrt{\rho^2 + L^2}}. \tag{9}$$

(c) Find the magnetic field at the center of a square loop, which carries a steady current I. Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2 + L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R}\frac{4}{\pi}\tan\frac{\pi}{4}. \tag{10}$$

(d) Show that the magnetic field at the center of a regular n-sided polygon, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R}\frac{n}{\pi}\tan\frac{\pi}{n}, \tag{11}$$

where R is the distance from center to a corner of the polygon.

(e) Show that the magnetic field at the center of a circular loop of radius R,

$$B = \frac{\mu_0 I}{2R}, \tag{12}$$

is obtained in the limit $n \to \infty$.