Homework No. 03 (2021 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Feb 9, 12.30pm

1. (**20 points.**) The vector potential for a point magnetic moment $\mathbf{m}$ is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{1}$$

Verify that the magnetic field due to the point dipole obtained by evaluating the curl

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{2}$$

can be expressed in the form

$$\mathbf{B}(\mathbf{r}) = \mathbf{m}\,\mu_0\,\delta^{(3)}(\mathbf{r}) + \frac{\mu_0}{4\pi}(\mathbf{m}\cdot\boldsymbol{\nabla})\left(\boldsymbol{\nabla}\frac{1}{r}\right). \tag{3}$$

Verify that the magnetic field satisfies the Maxwell equation

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{4}$$

Show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\Big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\Big] + \mathbf{m}\,\mu_0\,\delta^{(3)}(\mathbf{r}). \tag{5}$$

2. (**40 points.**) A charged spherical shell of radius a carries a total charge q uniformly distributed on the shell. It rotates with angular velocity $\boldsymbol{\omega}$ about a diameter.

   (a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{4\pi a^2}\,\boldsymbol{\omega}\times\mathbf{r}\,\delta(r-a). \tag{6}$$

   Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ for circular motion.

   (b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{7}$$

   determine the magnetic dipole moment of the rotating sphere to be

$$\mathbf{m} = \frac{qa^2}{3}\boldsymbol{\omega}. \tag{8}$$

(c) Evaluate the vector potential inside and outside the sphere to be

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi} \dfrac{\mathbf{m} \times \mathbf{r}}{a^3}, & r < a, \\ \dfrac{\mu_0}{4\pi} \dfrac{\mathbf{m} \times \mathbf{r}}{r^3}, & a < r. \end{cases} \tag{9}$$

Hint: Out of the three vectors $\boldsymbol{\omega}$, the observation point $\mathbf{r}$, and the integration variable $\mathbf{r}'$, choose $\mathbf{r}$ to be along the z axis while working in spherical polar coordinates. This leads to considerable simplification in the expression for $|\mathbf{r} - \mathbf{r}'|$ appearing in the denominator. Otherwise, without choosing $\mathbf{r}$ to be along $\hat{\mathbf{z}}$, use the ideas of Legendre polynomials and spherical harmonics.

(d) Derive the corresponding expression for the magnetic field, using $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, to be

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi} \dfrac{2\mathbf{m}}{a^3}, & r < a, \\ \dfrac{\mu_0}{4\pi} \dfrac{1}{r^3} \Big[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m} \Big], & a < r. \end{cases} \tag{10}$$