Homework No. 06 (2021 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Mar 16, 12.30pm

1. (**20 points.**) How does the operator
$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \tag{1}$$
in the wave equation transform under the Lorentz transformtion
$$z' = \gamma z + \beta\gamma ct, \tag{2a}$$
$$ct' = \beta\gamma z + \gamma ct. \tag{2b}$$

2. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are
$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{3}$$
respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are
$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4}$$
respectively. For infinitesimal transformations, $\beta_i = \delta\beta_i$ and $\omega_i = \delta\omega_i$ use the approximations
$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \delta\omega_i, \tag{5}$$
to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,
$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{6}$$
respectively. Then derive
$$\Big[N_1, N_2\Big] = N_1N_2 - N_2N_1 = J_3. \tag{7}$$
This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

(a) Is velocity addition commutative?

(b) Is velocity addition associative?

(c) Read a resource article (Wikipedia) on Wigner rotation.

3. (**60 points.**) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}, \tag{8}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [1].) An interactive applet for exploring velocity addition is available at Kocik's web page [2]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.)

(a) Analyse what is obtained if you add two subluminal speeds?

(b) Analyse what is obtained if you add a subluminal speed to speed of light?

(c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?

(d) Analyse what is obtained if you add speed of light to another speed of light?

(e) Analyse what is obtained if you add a superluminal speed to speed of light?

(f) Analyse what is obtained if you add two superluminal speeds?

References

[1] J. Kocik. Geometric diagram for relativistic addition of velocities. *Am. J. Phys.*, 80:737–739, August 2012.

[2] J. Kocik. An interactive applet for exploring relativistic velocity addition. http://www.mathoutlet.com/2016/08/relativistic-composition-of-velocities.html.