Homework No. 08 (2021 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2021 Apr 13, 12.30pm

1. (**20 points.**) Using Maxwell's equations, without introducing potentials, show that the electric and magnetic fields satisfy the inhomogeneous wave equations

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\boldsymbol{\nabla}\rho(\mathbf{r},t) - \frac{1}{\varepsilon_0}\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t), \tag{1a}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = \mu_0\boldsymbol{\nabla}\times\mathbf{J}(\mathbf{r},t). \tag{1b}$$

2. (**20 points.**) Consider the retarded Green's function

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right). \tag{2}$$

   (a) For $\mathbf{r}' = 0$ and $t' = 0$ show that

$$G(r,t) = \frac{1}{4\pi r}\,\delta\left(t-\frac{r}{c}\right). \tag{3}$$

   (b) Then, evaluate

$$\int_{-\infty}^{\infty} dt\, G(r,t). \tag{4}$$

   (c) From the answer above, what can you comment on the physical interpretation of $\int_{-\infty}^{\infty} dt\, G(r,t)$.

3. (**20 points.**) The n-dimensional Euclidean Green's function satisfies

$$-\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) G_E^{(n)}(x_1,\ldots,x_n) = \delta(x_1)\cdots\delta(x_n). \tag{5}$$

   (a) Show that the solution to this equation can be written as the Fourier transform

$$G_E^{(n)}(x_1,\ldots,x_n) = \int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{dk_n}{2\pi}\,\frac{e^{i(k_1x_1+\ldots+k_nx_n)}}{k_1^2+\ldots+k_n^2}. \tag{6}$$

   (b) Verify the integral

$$\frac{1}{M} = \int_0^{\infty} ds\, e^{-sM}. \tag{7}$$

(c) Using Eq. (7) in Eq. (6) show that

$$G_E^{(n)}(x_1,\ldots,x_n) = \int_0^\infty ds \prod_{m=1}^{n} \left[\int_{-\infty}^{\infty} \frac{dk_m}{2\pi} e^{-sk_m^2 + ik_m x_m} \right]. \tag{8}$$

(d) Show that

$$\int_{-\infty}^{\infty} dk_m e^{-sk_m^2 + ik_m x_m} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{x_m^2}{4s}} \tag{9}$$

(e) Substitute the integral of Eq. (9) in Eq. (8), and use the integral representation of Gamma function,

$$\Gamma(z) = \int_0^\infty \frac{dt}{t} t^z e^{-t}, \tag{10}$$

where $\Gamma(z)$ is the analytic continuation of factorial, $n! = \Gamma(n+1)$, after substituting $s = 1/t$ there, to show that

$$G_E^{(n)}(x_1,\ldots,x_n) = \left(\frac{\sqrt{\pi}}{2\pi}\right)^n \Gamma\left(\frac{n}{2} - 1\right) \left(\frac{4}{x_1^2 + \ldots + x_n^2}\right)^{\frac{n}{2}-1}. \tag{11}$$

(f) Verify that

$$G_E^{(3)} = \frac{1}{4\pi}\frac{1}{R_3} \tag{12}$$

and

$$G_E^{(4)} = \frac{1}{4\pi^2}\frac{1}{R_4^2}, \tag{13}$$

where $R_n^2 = x_1^2 + \ldots + x_n^2$.

(g) Show that integration of the Euclidean Green's function over one coordinate leads to the Euclidean Green's function in one lower dimension,

$$\int_{-\infty}^{\infty} dx_n G_E^{(n)}(x_1,\ldots,x_n) = G_E^{(n-1)}(x_1,\ldots,x_{n-1}). \tag{14}$$

Hint: Substitute $x_n = R_{n-1}\tan\theta$ and use the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta (\cos\theta)^{n-4} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{n}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{n}{2} - 1\right)}, \qquad \mathrm{Re}\, n > 3. \tag{15}$$

4. (**20 points.**) Evaluate the integral

$$\zeta(s) = \lim_{\epsilon\to 0+} \int_\epsilon^\infty dx \left(\frac{\pi}{x}\right)^s \delta(\sin x) \tag{16}$$

as a sum. The resultant sum is the Riemann zeta function. Determine $\zeta(2)$.
Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left| \frac{dF}{dx}\Big|_{x=a_r} \right|}, \tag{17}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$.