Homework No. 09 (2021 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Thursday, 2021 Apr 22, 12.30pm

1. (**20 points.**) Consider a particle of charge q moving along the path $\mathbf{r}_q(t)$. The corresponding charge density and current density are

$$\rho(\mathbf{r}', t') = q\,\delta^{(3)}(\mathbf{r}' - \mathbf{r}_q(t')), \tag{1a}$$

$$\mathbf{j}(\mathbf{r}', t') = q\,\mathbf{v}_q(t')\delta^{(3)}(\mathbf{r}' - \mathbf{r}_q(t')), \tag{1b}$$

where $\mathbf{v}_q(t)$ is the velocity of the particle at time t.

(a) Beginning from

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \int_{-\infty}^{\infty} dt'\, \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}\delta\left(t - t' - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right), \tag{2a}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{\infty} dt'\, \frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}\delta\left(t - t' - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right), \tag{2b}$$

and using Eqs. (1) derive

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} dt'\, \frac{\delta\left(t - t' - \frac{1}{c}|\mathbf{r} - \mathbf{r}_q(t')|\right)}{|\mathbf{r} - \mathbf{r}_q(t')|}, \tag{3a}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dt'\, q\mathbf{v}_q(t')\, \frac{\delta\left(t - t' - \frac{1}{c}|\mathbf{r} - \mathbf{r}_q(t')|\right)}{|\mathbf{r} - \mathbf{r}_q(t')|}. \tag{3b}$$

(b) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{4}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, evaluate the integrals (requiring the roots to be causal, that is, $t_r < t$) in Eqs. (3) as

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\left[|\mathbf{r} - \mathbf{r}(t_r)| - \frac{\mathbf{v}_q(t_r)}{c}\cdot\Big\{\mathbf{r} - \mathbf{r}_q(t_r)\Big\}\right]}, \tag{5a}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}_q(t')}{\left[|\mathbf{r} - \mathbf{r}(t_r)| - \frac{\mathbf{v}_q(t_r)}{c}\cdot\Big\{\mathbf{r} - \mathbf{r}_q(t_r)\Big\}\right]}, \tag{5b}$$

where t_r is uniquely determined using

$$F(t_r) = c(t - t_r) - |\mathbf{r} - \mathbf{r}(t_r)| = 0, \qquad t_r < t. \tag{6}$$

(c) In terms of the four-vectors

$$x^\alpha - x_q^\alpha(t_r) = (ct - ct_r, \mathbf{r} - \mathbf{r}_q(t_r)) \tag{7}$$

and

$$u_q^\alpha = \gamma_q(c, \mathbf{v}_q(t_r)), \qquad \gamma_q = \frac{1}{\sqrt{1 - \frac{\mathbf{v}_q(t_r)^2}{c^2}}}, \tag{8}$$

show that the expression in the denominator can be interpreted as

$$-\frac{1}{c\gamma_q}(u_q)_\alpha(x^\alpha - x_q^\alpha(t_r)) = c(t - t_r) - \frac{\mathbf{v}_q(t_r)}{c} \cdot \Big\{\mathbf{r} - \mathbf{r}_q(t_r)\Big\} \tag{9a}$$

$$= |\mathbf{r} - \mathbf{r}(t_r)| - \frac{\mathbf{v}_q(t_r)}{c} \cdot \Big\{\mathbf{r} - \mathbf{r}_q(t_r)\Big\}. \tag{9b}$$

Thus, $F(t_r) = 0$ implies

$$(u_q)_\alpha(x^\alpha - x_q^\alpha(t_r)) = 0, \tag{10}$$

stating that these events are separated by light-like distance.

2. (**20 points.**) A charged particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$. The electric and magnetic field generated by this charged particle is given by

$$\mathbf{E}(\mathbf{r}, t) = (1 - \beta^2)\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - vt)\hat{\mathbf{k}}}{[(x^2 + y^2)(1 - \beta^2) + (z - vt)^2]^{\frac{3}{2}}}, \tag{11a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \beta(1 - \beta^2)\frac{q}{4\pi\varepsilon_0}\frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{[(x^2 + y^2)(1 - \beta^2) + (z - vt)^2]^{\frac{3}{2}}}. \tag{11b}$$

Evaluate the electromagnetic momentum density for this configuration by evaluating

$$\mathbf{G}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t). \tag{12}$$

3. (**20 points.**) The electric and magnetic field generated by a particle with charge q moving along the z axis with speed v, $\beta = v/c$, can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{\big[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - vt)\hat{\mathbf{k}}\big]}{(x^2 + y^2)}\frac{(x^2 + y^2)(1 - \beta^2)}{[(x^2 + y^2)(1 - \beta^2) + (z - vt)^2]^{\frac{3}{2}}}, \tag{13a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\beta} \times \mathbf{E}(\mathbf{r}, t). \tag{13b}$$

(a) Consider the distribution

$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{2}\frac{\epsilon}{(x^2+\epsilon)^{\frac{3}{2}}}. \tag{14}$$

Show that

$$\delta(x) \begin{cases} \to \frac{1}{2\sqrt{\epsilon}} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon}{2x^3} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{15}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{16}$$

(b) Thus, verify that the electric and magnetic field of a charge approaching the speed of light can be expressed in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}\,\delta(z-ct), \tag{17a}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct) = 2q\left(\frac{\mu_0 c}{4\pi}\right)\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct), \tag{17b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\rho = \sqrt{x^2+y^2}$. These fields are confined on the $z = ct$ plane moving with speed c. Illustrate this configuration of fields using a diagram.

(c) To confirm that the above confined fields are indeed solutions to the Maxwell equations, verify the following:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0} q\delta^{(2)}(\boldsymbol{\rho})\delta(z-ct), \tag{18a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{18b}$$

$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \tag{18c}$$

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0 qc\hat{\mathbf{z}}\delta^{(2)}(\boldsymbol{\rho})\delta(z-ct). \tag{18d}$$

This is facilitated by writing

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{19}$$

and accomplished by using the following identities:

$$\boldsymbol{\nabla}_\rho\cdot\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho\times\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{20a}$$

$$\boldsymbol{\nabla}_\rho\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \boldsymbol{\nabla}_\rho\times\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{20b}$$