# (Preview of) Final Exam (Fall 2022)

## PHYS 500A: MATHEMATICAL METHODS

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1. (**20 points.**) Not available in preview mode.
2. (**20 points.**) Not available in preview mode.
3. (**20 points.**) Polynomials $(\mathbf{a}\cdot\mathbf{r})^l$ of degree $l$ satisfy the Laplacian when $\mathbf{a}$ is a null-vector, that is,

$$(\mathbf{a}\cdot\mathbf{a}) = 0. \tag{1}$$

(a) Show that

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = l(l-1)(\mathbf{a}\cdot\mathbf{r})^{(l-2)}(\mathbf{a}\cdot\mathbf{a}), \tag{2}$$

and conclude

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = 0. \tag{3}$$

(b) Write the polynomial construction in the form

$$(\mathbf{a}\cdot\mathbf{r})^l = r^l(\mathbf{a}\cdot\hat{\mathbf{r}})^l. \tag{4}$$

Observe that $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ has no radial dependence. Thus, in this form, the radial and angular dependence is separated. Starting from the Laplacian in spherical polar coordinates,

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\mathbf{r})^l = 0, \tag{5}$$

deduce

$$\frac{r^l}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l + (\mathbf{a}\cdot\hat{\mathbf{r}})^l\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l = 0. \tag{6}$$

(c) Show that

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l = l(l+1)\frac{r^l}{r^2}. \tag{7}$$

Thus, derive the differential equation for the generating function

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l = 0. \tag{8}$$

(d) Use the generating function

$$\frac{(\mathbf{a}\cdot\hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l} \psi_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta,\phi) \tag{9}$$

written in terms of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{10}$$

to derive

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] Y_{lm}(\theta,\phi) = 0. \tag{11}$$

4. (**20 points.**) An example of a null-vector is

$$\mathbf{a} = (-i\cos\alpha, -i\sin\alpha, 1). \tag{12}$$

(a) Identify the corresponding $y_\pm$ to show that, now, $\psi_{lm}$ in the generating function is

$$\psi_{lm} = \frac{e^{-im\left(\alpha-\frac{\pi}{2}\right)}}{\sqrt{(l+m)!(l-m)!}}. \tag{13}$$

(b) Then, integrate to derive an integral representation for spherical harmonics,

$$\frac{1}{l!}\int_0^{2\pi} \frac{d\alpha}{2\pi} e^{im\alpha} \big[\cos\theta - i\sin\theta\cos(\phi-\alpha)\big]^l = \sqrt{\frac{4\pi}{2l+1}} \frac{i^m\, Y_{lm}(\theta,\phi)}{\sqrt{(l+m)!(l-m)!}}. \tag{14}$$

(c) By setting $m = 0$ derive the corresponding integral representation for Legendre polynomial $P_l(\cos\theta)$:

$$\int_0^{\pi} \frac{d\alpha}{\pi} \big[\cos\theta - i\sin\theta\cos\alpha\big]^l = P_l(\cos\theta). \tag{15}$$

5. (**20 points.**) For a null-vector $\mathbf{a}$, that satisfies

$$\mathbf{a}\cdot\mathbf{a} = 0, \tag{16}$$

the polynomial $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ of degree $l$ is the generating function of spherical harmonics $Y_{lm}(\theta,\phi)$. To derive the orthonormality properties of spherical harmonics let us consider the product of two generating functions, with null-vectors $\mathbf{a}$ and $\mathbf{a}^*$, integrated over all the angles,

$$\int d\Omega\, (\mathbf{a}^*\cdot\hat{\mathbf{r}})^l (\mathbf{a}\cdot\hat{\mathbf{r}})^{l'}, \tag{17}$$

where

$$d\Omega = \sin\theta d\theta d\phi. \tag{18}$$

(a) After integration over the angles the product of the two generating functions is a scalar. Thus, it has to be constructed out of $(\mathbf{a}\cdot\mathbf{a})$, $(\mathbf{a}^*\cdot\mathbf{a}^*)$, and $(\mathbf{a}^*\cdot\mathbf{a})$. Since $(\mathbf{a}\cdot\mathbf{a}) = 0$ and $(\mathbf{a}^*\cdot\mathbf{a}^*) = 0$, the integral has to be constructed out of $(\mathbf{a}^*\cdot\mathbf{a})$. This is possible only if $l = l'$. Together, we conclude

$$\int d\Omega\,(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'} = \delta_{ll'}(\mathbf{a}^*\cdot\mathbf{a})^l C_l, \tag{19}$$

in terms of arbitrary constant $C_l$.

(b) To determine $C_l$ choose

$$\mathbf{a} = (1, i, 0). \tag{20}$$

For this choice of null-vector, evaluate $\mathbf{a}^* = (1,-i,0)$, $(\mathbf{a}\cdot\hat{\mathbf{r}}) = \sin\theta e^{i\phi}$, $(\mathbf{a}^*\cdot\hat{\mathbf{r}}) = \sin\theta e^{-i\phi}$, and $(\mathbf{a}^*\cdot\hat{\mathbf{a}}) = 2$. Thus, find

$$C_l = \frac{4\pi}{2^l}\int_0^1 dt(1-t^2)^l, \tag{21}$$

after substituting $\cos\theta = t$. Evaluate

$$C_0 = 4\pi. \tag{22}$$

Integrate by parts in the integral for $C_l$ to derive the recurrence relation

$$C_l = \frac{l}{2l+1}C_{l-1}. \tag{23}$$

Evaluate

$$C_l = \frac{4\pi 2^l l! l!}{(2l+1)!}. \tag{24}$$

Thus, conclude

$$\int d\Omega\,\frac{(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l}{l!}\frac{(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'}}{l!} = \delta_{ll'}4\pi\frac{(\mathbf{a}^*\cdot\mathbf{a})^l 2^l}{(2l+1)!}. \tag{25}$$

(c) For null-vectors constructed out of $y_\pm$ in the form

$$\mathbf{a} = \left(\frac{y_-^2 - y_+^2}{2}, \frac{y_-^2 + y_+^2}{2i}, y_+y_-\right) \tag{26}$$

show that

$$4\pi\frac{(\mathbf{a}^*\cdot\mathbf{a})^l 2^l}{(2l+1)!} = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\psi_{lm}^*\psi_{l'm'}\delta_{mm'}, \tag{27}$$

where

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}}. \tag{28}$$

Using the generating function

$$\frac{(\mathbf{a}^* \cdot \hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l} \psi_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) \tag{29}$$

show that

$$\begin{aligned}\sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \psi^*_{lm} \psi_{l'm'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \int d\Omega\, Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) \\ = \delta_{ll'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \psi^*_{lm} \psi_{l'm'} \delta_{mm'}.\end{aligned} \tag{30}$$

Thus, comparing the two sides of the equality, read out the orthonormality condition for the spherical harmonics,

$$\int d\Omega\, Y^*_{lm}(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}. \tag{31}$$