Final Exam (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

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1. (**20 points.**) Given
$$\left(\frac{a}{r}+\frac{\partial}{\partial r}\right)\left(\frac{b}{r}+\frac{\partial}{\partial r}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}. \tag{1}$$
Find the numbers a and b.

2. (**20 points.**) Given
$$\nabla^2(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})=c. \tag{2}$$
Find the scalar c.

3. (**20 points.**) Polynomials $(\mathbf{a}\cdot\mathbf{r})^l$ of degree l satisfy the Laplacian when $\mathbf{a}$ is a null-vector, that is,
$$(\mathbf{a}\cdot\mathbf{a})=0. \tag{3}$$
   (a) Show that
$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l=l(l-1)(\mathbf{a}\cdot\mathbf{r})^{(l-2)}(\mathbf{a}\cdot\mathbf{a}), \tag{4}$$
   and conclude
$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l=0. \tag{5}$$
   (b) Write the polynomial construction in the form
$$(\mathbf{a}\cdot\mathbf{r})^l=r^l(\mathbf{a}\cdot\hat{\mathbf{r}})^l. \tag{6}$$
   Observe that $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ has no radial dependence. Thus, in this form, the radial and angular dependence is separated. Starting from the Laplacian in spherical polar coordinates,
$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\mathbf{r})^l=0, \tag{7}$$
   deduce
$$\frac{r^l}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l+(\mathbf{a}\cdot\hat{\mathbf{r}})^l\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l=0. \tag{8}$$

(c) Show that

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l = l(l+1)\frac{r^l}{r^2}. \tag{9}$$

Thus, derive the differential equation for the generating function

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l = 0. \tag{10}$$

(d) Use the generating function

$$\frac{(\mathbf{a}\cdot\hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l}\psi_{lm}\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi) \tag{11}$$

written in terms of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{12}$$

to derive

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right]Y_{lm}(\theta,\phi) = 0. \tag{13}$$

4. (**20 points.**) An example of a null-vector is

$$\mathbf{a} = (-i\cos\alpha, -i\sin\alpha, 1). \tag{14}$$

(a) Identify the corresponding $y_\pm$ to show that, now, ψ_{lm} in the generating function is

$$\psi_{lm} = \frac{e^{-im\left(\alpha-\frac{\pi}{2}\right)}}{\sqrt{(l+m)!(l-m)!}}. \tag{15}$$

(b) Then, integrate to derive an integral representation for spherical harmonics,

$$\frac{1}{l!}\int_0^{2\pi}\frac{d\alpha}{2\pi}\,e^{im\alpha}\big[\cos\theta - i\sin\theta\cos(\phi-\alpha)\big]^l = \sqrt{\frac{4\pi}{2l+1}}\frac{i^m\,Y_{lm}(\theta,\phi)}{\sqrt{(l+m)!(l-m)!}}. \tag{16}$$

(c) By setting $m = 0$ derive the corresponding integral representation for Legendre polynomial $P_l(\cos\theta)$:

$$\int_0^{\pi}\frac{d\alpha}{\pi}\big[\cos\theta - i\sin\theta\cos\alpha\big]^l = P_l(\cos\theta). \tag{17}$$

5. (**20 points.**) For a null-vector $\mathbf{a}$, that satisfies

$$\mathbf{a}\cdot\mathbf{a} = 0, \tag{18}$$

the polynomial $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ of degree l is the generating function of spherical harmonics $Y_{lm}(\theta,\phi)$. To derive the orthonormality properties of spherical harmonics let us consider the product of two generating functions, with null-vectors $\mathbf{a}$ and $\mathbf{a}^*$, integrated over all the angles,

$$\int d\Omega\,(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'}, \tag{19}$$

where

$$d\Omega = \sin\theta d\theta d\phi. \tag{20}$$

(a) After integration over the angles the product of the two generating functions is a scalar. Thus, it has to be constructed out of $(\mathbf{a}\cdot\mathbf{a})$, $(\mathbf{a}^*\cdot\mathbf{a}^*)$, and $(\mathbf{a}^*\cdot\mathbf{a})$. Since $(\mathbf{a}\cdot\mathbf{a})=0$ and $(\mathbf{a}^*\cdot\mathbf{a}^*)=0$, the integral has to be constructed out of $(\mathbf{a}^*\cdot\mathbf{a})$. This is possible only if $l=l'$. Together, we conclude

$$\int d\Omega\,(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'} = \delta_{ll'}(\mathbf{a}^*\cdot\mathbf{a})^l C_l, \tag{21}$$

in terms of arbitrary constant C_l.

(b) To determine C_l choose

$$\mathbf{a} = (1, i, 0). \tag{22}$$

For this choice of null-vector, evaluate $\mathbf{a}^* = (1,-i,0)$, $(\mathbf{a}\cdot\hat{\mathbf{r}}) = \sin\theta e^{i\phi}$, $(\mathbf{a}^*\cdot\hat{\mathbf{r}}) = \sin\theta e^{-i\phi}$, and $(\mathbf{a}^*\cdot\hat{\mathbf{a}}) = 2$. Thus, find

$$C_l = \frac{4\pi}{2^l}\int_0^1 dt(1-t^2)^l, \tag{23}$$

after substituting $\cos\theta = t$. Evaluate

$$C_0 = 4\pi. \tag{24}$$

Integrate by parts in the integral for C_l to derive the recurrence relation

$$C_l = \frac{l}{2l+1}C_{l-1}. \tag{25}$$

Evaluate

$$C_l = \frac{4\pi 2^l l! l!}{(2l+1)!}. \tag{26}$$

Thus, conclude

$$\int d\Omega\,\frac{(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l}{l!}\frac{(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'}}{l!} = \delta_{ll'}4\pi\frac{(\mathbf{a}^*\cdot\mathbf{a})^l 2^l}{(2l+1)!}. \tag{27}$$

(c) For null-vectors constructed out of $y_\pm$ in the form

$$\mathbf{a} = \left(\frac{y_-^2 - y_+^2}{2}, \frac{y_-^2 + y_+^2}{2i}, y_+y_-\right) \tag{28}$$

show that

$$4\pi\frac{(\mathbf{a}^*\cdot\mathbf{a})^l 2^l}{(2l+1)!} = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\psi^*_{lm}\psi_{l'm'}\delta_{mm'}, \tag{29}$$

where

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}}. \tag{30}$$

Using the generating function

$$\frac{(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l}\psi_{lm}\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi) \tag{31}$$

show that

$$\begin{aligned}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\psi^*_{lm}\psi_{l'm'}\sqrt{\frac{4\pi}{2l+1}}\sqrt{\frac{4\pi}{2l'+1}}\int d\Omega\, Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi)\\ = \delta_{ll'}\sqrt{\frac{4\pi}{2l+1}}\sqrt{\frac{4\pi}{2l'+1}}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\psi^*_{lm}\psi_{l'm'}\delta_{mm'}.\end{aligned} \tag{32}$$

Thus, comparing the two sides of the equality, read out the orthonormality condition for the spherical harmonics,

$$\int d\Omega\, Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{33}$$