Midterm Exam No. 01 (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

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Date: 2022 Sep 30

1. (**20 points.**) Using the property of Kronecker δ-function and Levi-Civita symbol evaluate the following using index notation,

$$\varepsilon_{ijk}\delta_{im}\delta_{jn}\delta_{ij}. \tag{1}$$

2. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}\left(\frac{1}{\mathbf{r}\cdot\mathbf{p}}\right) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{2}$$

where $\mathbf{p}$ is a constant vector. Thus, find a and b.

3. (**20 points.**) Given

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{3a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{3b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{3c}$$

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}, \tag{3d}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{k}}. \tag{3e}$$

and the relation

$$\hat{\boldsymbol{\rho}} = a\,\hat{\mathbf{r}} + b\,\hat{\boldsymbol{\theta}} + c\,\hat{\boldsymbol{\phi}}. \tag{4}$$

Find the components a, b, and c, such that the above equation is an identity.

4. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\epsilon\to 0}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}\frac{2}{\pi}. \tag{5}$$

Show that

$$\delta(x) = \begin{cases} \to \frac{1}{\epsilon} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon^3}{x^4} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{6}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\,\delta(x) = 1. \tag{7}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

5. (**10 points.**) The tangent and normal vectors for the cylindrical coordinate system are

$$\mathbf{e}_1 = \mathbf{e}_\rho = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^1 = \mathbf{e}^\rho = \hat{\boldsymbol{\rho}}, \tag{8a}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}^2 = \mathbf{e}^\phi = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \tag{8b}$$

$$\mathbf{e}_3 = \mathbf{e}_z = \hat{\mathbf{z}}, \qquad \mathbf{e}^3 = \mathbf{e}^z = \hat{\mathbf{z}}. \tag{8c}$$

Compute the Christoffel symbol

$$\mathbf{e}_2 \cdot (\boldsymbol{\nabla}\mathbf{e}_1) \cdot \mathbf{e}^2. \tag{9}$$