Midterm Exam No. 02 (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

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1. (**20 points.**) Given

$$y(x) = \int_x^0 ds\, f(s). \tag{1}$$

Evaluate dy/dx.

2. (**20 points.**) Find the solution to the linear differential equation

$$\left[\frac{d^3}{dt^3} + 3\frac{d^2}{dt^2} + 3\frac{d}{dt} + 1\right] x(t) = 0 \tag{2}$$

for initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, and $\ddot{x}(0) = a_0$.

3. (**20 points.**) Verify the identity

$$\phi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\psi) - \psi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\phi) = \boldsymbol{\nabla}\cdot[\lambda(\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi)], \tag{3}$$

which is a slight generalization of what is known as Green's second identity. Here ϕ, ψ, and λ, are position dependent functions.

4. (**20 points.**) The expression for the electric potential due to a point charge placed in front of a perfectly conducting semi-infinite slab, described by

$$\frac{\varepsilon(z)}{\varepsilon_0} = \begin{cases} \infty, & z < 0, \\ 1, & 0 < z, \end{cases} \tag{4}$$

is given in terms of the reduced Green function that satisfies the differential equation $(0 < \{z, z'\})$

$$-\left[\frac{\partial^2}{\partial z^2} - k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{5}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and at $z \to \infty$.

(a) Construct the reduced Green function in the form

$$\varepsilon_0 g(z, z') = \begin{cases} Ae^{kz} + Be^{-kz}, & 0 < z < z', \\ Ce^{kz} + De^{-kz}, & 0 < z' < z, \end{cases} \tag{6}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$\begin{aligned}\varepsilon_0 g(0, z') &= 0, & \text{(7a)}\\ \varepsilon_0 g(\infty, z') &= 0, & \text{(7b)}\\ \varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} &= 0, & \text{(7c)}\\ \partial_z \varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} &= -1. & \text{(7d)}\end{aligned}$$

(b) Express the solution in the form

$$\varepsilon_0 g(z, z') = \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{8}$$

5. (**20 points.**) The expression for the electric potential due to a point charge placed in between two parallel grounded perfectly conducting semi-infinite slabs, described by

$$\frac{\varepsilon(z)}{\varepsilon_0} = \begin{cases} \infty, & z < 0, \\ 1, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{9}$$

is given in terms of the reduced Green function that satisfies the differential equation $(0 < \{z, z'\} < a)$

$$\left[-\frac{\partial^2}{\partial z^2} + k^2 \right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{10}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$\varepsilon_0 g(z, z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{11}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$\begin{aligned}\varepsilon_0 g(0, z') &= 0, & \text{(12a)}\\ \varepsilon_0 g(a, z') &= 0, & \text{(12b)}\\ \varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} &= 0, & \text{(12c)}\\ \partial_z \varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} &= -1. & \text{(12d)}\end{aligned}$$

(b) After using conditions in Eqs. (12a) and (12b) show that the reduced Green's function can be expressed in the form

$$\varepsilon_0 g(z, z') = \begin{cases} A \sinh kz, & 0 < z < z' < a, \\ C' \sinh k(a - z), & 0 < z' < z < a, \end{cases} \tag{13}$$

where $C' = -C/\cosh ka$. Then, use Eqs. (12c) and (12d) to show that

$$\varepsilon_0 g(z, z') = \begin{cases} \dfrac{\sinh kz \sinh k(a - z')}{k \sinh ka}, & 0 < z < z' < a, \\ \dfrac{\sinh kz' \sinh k(a - z)}{k \sinh ka}, & 0 < z' < z < a. \end{cases} \tag{14}$$

(c) Take the limit $ka \to \infty$ in your solution above, (which corresponds to moving the slab at $z = a$ to infinity,) to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka \to \infty} \varepsilon_0 g(z, z') = \frac{1}{2k} e^{-k|z - z'|} - \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{15}$$

This should serve as a check for your solution to the reduced Green's function. Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{16}$$