Homework No. 02 (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2022 Sep 9, 4.30pm

1. (**20 points.**) Verify the following identities:

$$\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}, \tag{1a}$$

$$\boldsymbol{\nabla}\, \mathbf{r} = \mathbf{1}. \tag{1b}$$

   Further, show that

$$\boldsymbol{\nabla} \cdot \mathbf{r} = 3, \tag{2a}$$

$$\boldsymbol{\nabla} \times \mathbf{r} = 0. \tag{2b}$$

   Here r is the magnitude of the position vector $\mathbf{r}$, and $\hat{\mathbf{r}}$ is the unit vector pointing in the direction of $\mathbf{r}$.

2. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{p}) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{3}$$

   where $\mathbf{p}$ is a constant vector. Thus, find a and b.

3. (**20 points.**) Evaluate

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3} \right), \tag{4}$$

   everywhere in space, including $\mathbf{r} = 0$.
   Hint: Check your answer for consistency by using divergence theorem.

4. (**20 points.**) Consider the distribution

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}. \tag{5}$$

   Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{6}$$

   Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{7}$$

   Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

5. (**20 points.**) Consider the distribution

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}. \tag{8}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{9}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{10}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.