Homework No. 04 (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2022 Sep 30, 4.30pm

1. (**20 points.**) Consider the eigenvalue equation

$$\sigma_x|\sigma'_x\rangle = \sigma'_x|\sigma'_x\rangle, \tag{1}$$

where primes denote eigenvalues.

(a) Find the eigenvalues and normalized eigenvectors (up to a phase) of

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

For reference we shall call these eigenvectors $|\sigma'_x = +\rangle$ and $|\sigma'_x = -\rangle$.

(b) Now compute the new matrix

$$\bar{\sigma}_x = \begin{pmatrix} \langle\sigma'_x = +|\sigma_x|\sigma'_x = +\rangle & \langle\sigma'_x = +|\sigma_x|\sigma'_x = -\rangle \\ \langle\sigma'_x = -|\sigma_x|\sigma'_x = +\rangle & \langle\sigma'_x = -|\sigma_x|\sigma'_x = -\rangle \end{pmatrix}. \tag{3}$$

(c) Similarly, compute the new matrices

$$\bar{\sigma}_y = \begin{pmatrix} \langle\sigma'_x = +|\sigma_y|\sigma'_x = +\rangle & \langle\sigma'_x = +|\sigma_y|\sigma'_x = -\rangle \\ \langle\sigma'_x = -|\sigma_y|\sigma'_x = +\rangle & \langle\sigma'_x = -|\sigma_y|\sigma'_x = -\rangle \end{pmatrix}, \quad \text{where} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{4}$$

and

$$\bar{\sigma}_z = \begin{pmatrix} \langle\sigma'_x = +|\sigma_z|\sigma'_x = +\rangle & \langle\sigma'_x = +|\sigma_z|\sigma'_x = -\rangle \\ \langle\sigma'_x = -|\sigma_z|\sigma'_x = +\rangle & \langle\sigma'_x = -|\sigma_z|\sigma'_x = -\rangle \end{pmatrix}, \quad \text{where} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

(d) Find the product of the last two matrices, $\bar{\sigma}_y\bar{\sigma}_z$, and express it in terms of $\bar{\sigma}_x$.

2. (**20 points.**) In the eigenbasis,

$$|\sigma'_y = +\rangle = \frac{e^{i\alpha_1}}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{and} \quad |\sigma'_y = -\rangle = \frac{e^{i\alpha_2}}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, \tag{6}$$

of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{7}$$

where we included the arbitrariness in the phases as α_1 and α_2, the Pauli matrices are

$$\bar{\sigma}_x = \begin{pmatrix} 0 & -ie^{-i(\alpha_1-\alpha_2)} \\ ie^{i(\alpha_1-\alpha_2)} & 0 \end{pmatrix}, \tag{8a}$$

$$\bar{\sigma}_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{8b}$$

$$\bar{\sigma}_z = \begin{pmatrix} 0 & e^{-i(\alpha_1-\alpha_2)} \\ e^{i(\alpha_1-\alpha_2)} & 0 \end{pmatrix}. \tag{8c}$$

Thus, these representations of Pauli matrices depend on the arbitrary choice of phases. Do we loose the ability to predict the outcome of an experiment due to this arbitrariness? Verify that

$$\bar{\sigma}_x\bar{\sigma}_y = i\bar{\sigma}_z, \tag{9a}$$

$$\bar{\sigma}_y\bar{\sigma}_z = i\bar{\sigma}_x, \tag{9b}$$

$$\bar{\sigma}_z\bar{\sigma}_x = i\bar{\sigma}_y. \tag{9c}$$

Thus, the algebra of the Pauli matrices is independent of the arbitrariness in the phases. This will ensure that no measurable quantity depends on the choice of phases.

3. (**20 points.**) Construct the matrix

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}, \tag{10}$$

where

$$\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{i}} + \sigma_y\hat{\mathbf{j}} + \sigma_z\hat{\mathbf{k}}, \tag{11}$$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}. \tag{12}$$

Use the following representation of Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

Find the eigenvalues of the matrix $\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}$.

4. (**20 points.**) The Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

is written in the eigenbasis of

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Write σ_x in the eigenbasis of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{16}$$

Note that this representation has the arbitraryness of the choice of phase in the eigenvectors.

5. (**20 points.**) Watch the two-hour long discussion in the following YouTube video

   https://youtu.be/w6fe19FEaEI

   titled 'Quantum Mechanics: Measurement disturbs the system'. Submit a half-page brief on the topic of discussion.