

# Homework No. 05 (Fall 2022)

## PHYS 500A: MATHEMATICAL METHODS

*School of Physics and Applied Physics, Southern Illinois University–Carbondale*

Due date: Monday, 2022 Oct 10, 4.30pm

0. Problems 3, 4, 5, and 9, are for submission. Rest are for practice.

1. (20 points.) A damped harmonic oscillator, constituting of a body of mass  $m$  and a spring of spring constant  $k$ , is described by

$$ma = -kx - bv, \quad (1)$$

where  $x$  is position,  $v = dx/dt$  is velocity,  $a = dv/dt$  is acceleration, and  $b$  is the damping coefficient. Thus, we have the differential equation

$$\left[ \frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} + \omega_0^2 \right] x(t) = 0 \quad (2)$$

with initial conditions

$$x(0) = x_0, \quad (3a)$$

$$\dot{x}(0) = v_0, \quad (3b)$$

where

$$\omega_0^2 = \frac{k}{m}, \quad 2\gamma = \frac{b}{m}. \quad (4)$$

(a)  $\gamma = 0$ : In the absence of damping show that the solution is

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t. \quad (5)$$

(b)  $\gamma < \omega_0$ : Underdamped harmonic oscillator.

$$x(t) = e^{-\gamma t} \left[ x_0 \cos \sqrt{\omega_0^2 - \gamma^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t \right]. \quad (6)$$

(c)  $\gamma = \omega_0$ : Critically damped harmonic oscillator.

$$x(t) = e^{-\omega_0 t} [x_0 + (v_0 + \omega_0 x_0)t]. \quad (7)$$

(d)  $\gamma > \omega_0$ : Overdamped harmonic oscillator.

$$x(t) = e^{-\gamma t} \left[ x_0 \cosh \sqrt{\gamma^2 - \omega_0^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh \sqrt{\gamma^2 - \omega_0^2} t \right]. \quad (8)$$

(e) Set  $\omega_0 = 1$ , which is equivalent to the substitution  $\omega_0 t = \tau$ , and sets the scale for the time  $t$ . That is, time is measured in units of  $T = 2\pi/\omega_0$ . The system is then completely characterized by the parameter  $\gamma/\omega_0$  and the initial conditions  $x_0$  and  $v_0$ . Plot the solutions for the initial conditions  $x_0 = 0$  and  $v_0 = 1$ .

2. **(20 points.)** Starting from the solution for the position of an underdamped harmonic oscillator ( $\gamma < \omega_0$ ),

$$x(t) = e^{-\gamma t} \left[ x_0 \cos \sqrt{\omega_0^2 - \gamma^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t \right], \quad (9)$$

obtain the solution for the velocity  $v(t) = dx/dt$  of an underdamped harmonic oscillator ( $\gamma < \omega_0$ ) in the form

$$v(t) = e^{-\gamma t} \left[ v_0 \cos \sqrt{\omega_0^2 - \gamma^2} t - \frac{(\omega_0^2 x_0 + \gamma v_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t \right]. \quad (10)$$

3. **(20 points.)** A critically damped harmonic oscillator is described by the differential equation

$$\left[ \frac{d^2}{dt^2} + 2\omega_0 \frac{d}{dt} + \omega_0^2 \right] x(t) = 0, \quad (11)$$

where  $\omega_0$  is a characteristic frequency. Find the solution  $x(t)$  for initial conditions  $x(0) = x_0$  and  $\dot{x}(0) = 0$ . Plot  $x(t)$  as a function of  $t$  in the following graph where  $x_0 e^{-\omega_0 t}$  is already plotted for reference. For what  $t$  is the solution  $x(t)$  a maximum?

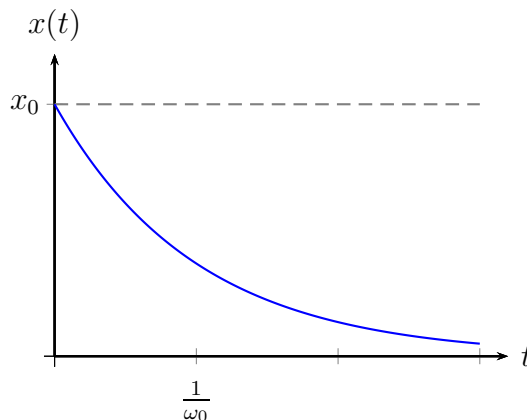


Figure 1: Critically damped harmonic oscillator.

4. **(20 points.)** A critically damped harmonic oscillator is described by the differential equation

$$\left[ \frac{d^2}{dt^2} + 2\omega_0 \frac{d}{dt} + \omega_0^2 \right] x(t) = 0, \quad (12)$$

where  $\omega_0$  is a characteristic frequency. Find the solution  $x(t)$  for initial conditions  $x(0) = 0$  and  $\dot{x}(0) = v_0$ . Plot  $x(t)$  as a function of  $t$  in the graph in Figure 2, where  $\omega_0$  and  $v_0/\omega_0$  is used to set scales for time  $t$  and position  $x(t)$ . For what  $t$  is the solution  $x(t)$  a maximum?

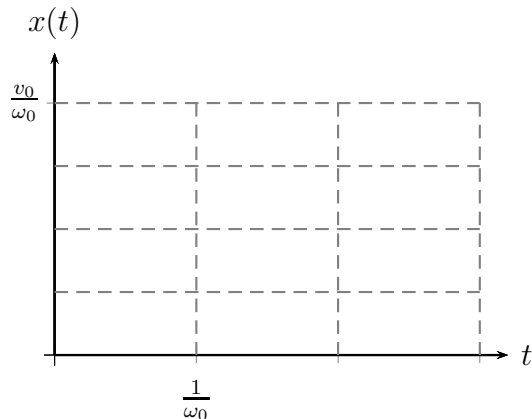


Figure 2: Critically damped harmonic oscillator.

5. (20 points.) A body experiencing only damping is described by the differential equation

$$\left[ \frac{d^2}{dt^2} + 2\gamma \frac{d}{dt} \right] x(t) = 0, \quad (13)$$

where  $\gamma$  is a measure of the damping. Find the solution  $x(t)$  for initial conditions  $x(0) = x_0$  and  $\dot{x}(0) = v_0$  to be

$$x(t) = x_0 + \frac{v_0}{2\gamma} [1 - e^{-2\gamma t}]. \quad (14)$$

Obtain the above expression starting from the solution for the overdamped harmonic oscillator ( $\gamma > \omega_0$ )

$$x(t) = e^{-\gamma t} \left[ x_0 \cosh \sqrt{\gamma^2 - \omega_0^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh \sqrt{\gamma^2 - \omega_0^2} t \right] \quad (15)$$

by setting  $\omega_0 = 0$ . Interpret the solution for  $v_0 = 0$ , why isn't there no motion?

6. (20 points.) Starting from the solution for the underdamped harmonic oscillator ( $\gamma < \omega_0$ ),

$$x(t) = e^{-\gamma t} \left[ x_0 \cos \sqrt{\omega_0^2 - \gamma^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t \right], \quad (16)$$

obtain the solution for the overdamped harmonic oscillator ( $\gamma > \omega_0$ ),

$$x(t) = e^{-\gamma t} \left[ x_0 \cosh \sqrt{\gamma^2 - \omega_0^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh \sqrt{\gamma^2 - \omega_0^2} t \right]. \quad (17)$$

7. **(20 points.)** Starting from the solution for the underdamped harmonic oscillator ( $\gamma < \omega_0$ ),

$$x(t) = e^{-\gamma t} \left[ x_0 \cos \sqrt{\omega_0^2 - \gamma^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t \right], \quad (18)$$

obtain the solution for the critically damped harmonic oscillator ( $\gamma = \omega_0$ ),

$$x(t) = e^{-\omega_0 t} [x_0 + (v_0 + \omega_0 x_0)t]. \quad (19)$$

8. **(20 points.)** The solution for the underdamped harmonic oscillator ( $\gamma < \omega_0$ ) is

$$x(t) = e^{-\gamma t} \left[ x_0 \cos \sqrt{\omega_0^2 - \gamma^2} t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t \right]. \quad (20)$$

For the initial condition  $x_0 = 0$  we have

$$x(t) = \frac{v_0 e^{-\gamma t}}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2} t. \quad (21)$$

Verify that the function

$$\frac{v_0 e^{-\gamma t}}{\sqrt{\omega_0^2 - \gamma^2}} \quad (22)$$

is an envelope to the solution  $x(t)$ . Investigate if this is an envelope for the case  $x_0 \neq 0$ .

9. **(20 points.)** Read the article titled ‘Life at low Reynolds number’ by E. M. Purcell, American Journal of Physics 45 (1977) 3. Here is the link to the article:

<http://dx.doi.org/10.1119/1.10903>

Here is a question asked to verify the understanding of the concept being discussed in the paper. Imagine a micrometer sized bacteria, shaped like a human, swimming in water using the methods used by a typical human swimmer. Qualitatively describe the motion of this hypothetical bacteria.