Homework No. 06 (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2022 Oct 17, 4.30pm

1. (**20 points.**) The Fourier space is spanned by the Fourier eigenfunctions

$$e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \ldots, \qquad 0 \le \phi < 2\pi. \tag{1}$$

An arbitrary function $f(\phi)$ has the Fourier series representation

$$f(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} a_m e^{im\phi}, \tag{2}$$

where $e^{im\phi}$ are the Fourier eigenfunctions and a_m are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{-in\phi} e^{im\phi} = \delta_{mn}. \tag{3}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$a_m = \int_0^{2\pi} d\phi \, e^{-im\phi} f(\phi). \tag{4}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} e^{-im\phi'} = \delta(\phi - \phi'). \tag{5}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{d\phi^2} - m^2 \right] e^{im\phi} = 0. \tag{6}$$

(e) Green's function: The associated Green's function satisfies the equation

$$-\left[\frac{d^2}{d\phi^2} - m^2 \right] g(\phi, \phi') = \delta(\phi - \phi'). \tag{7}$$

To determine the Fourier components of $\tan\phi$ start from

$$\tan\phi = \frac{1}{i}\frac{e^{i\phi}-e^{-i\phi}}{e^{i\phi}+e^{-i\phi}} \tag{8}$$

and show that

$$\tan\phi = \frac{1}{i} + \sum_{m=1}^{\infty} e^{-2im\phi}\frac{2(-1)^m}{i}. \tag{9}$$

Thus, read out all the Fourier components. Similarly, find the Fourier components of $\cot\phi$.

2. (**20 points.**) The (continuous) Fourier space is spanned by the Fourier eigenfunctions

$$e^{ikx}, \qquad -\infty < k < \infty, \qquad -\infty < x < \infty. \tag{10}$$

An arbitrary function $f(x)$ has the Fourier series representation

$$f(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\tilde{f}(k), \tag{11}$$

where e^{ikx} are the Fourier eigenfunctions and $\tilde{f}(k)$ are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\, e^{-ik'x}e^{ikx} = \delta(k-k'). \tag{12}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty}dx\, e^{-ikx}f(x). \tag{13}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}e^{-ikx'} = \delta(x-x'). \tag{14}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{dx^2} - k^2\right]e^{ikx} = 0. \tag{15}$$

Consider the inhomogeneous linear differential equation

$$\left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right)f(x) = \delta(x). \tag{16}$$

Use the Fourier transformation and the associated inverse Fourier transformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{17a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \tag{17b}$$

to show that the corresponding equation satisfied by $\tilde{f}(k)$ is algebraic. Find $\tilde{f}(k)$.

3. (**20 points.**) The half-range Fourier space is spanned by the Fourier eigenfunctions

$$\sin m\phi, \qquad m = 1, 2, 3, \ldots, \qquad 0 \leq \phi \leq \pi. \tag{18}$$

An arbitrary function $f(\phi)$, for ϕ limited to half the range, has the half-range Fourier series representation

$$f(\phi) = \sum_{m=1}^{\infty} a_m \sin m\phi, \tag{19}$$

where $\sin m\phi$ are the half-range Fourier eigenfunctions and a_m are the respective half-range Fourier components.

(a) Orthogonality relation: The half-range Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{2}{\pi} \int_0^{\pi} d\phi \, \sin m\phi \, \sin m'\phi = \delta_{mm'}. \tag{20}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$a_m = \frac{2}{\pi} \int_0^{\pi} d\phi \, \sin m\phi \, f(\phi). \tag{21}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\frac{2}{\pi} \sum_{m=1}^{\infty} \sin m\phi \, \sin m\phi' = \delta(\phi - \phi'). \tag{22}$$

(d) Differential equation: The half-range Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{d\phi^2} - m^2\right] \sin m\phi = 0. \tag{23}$$

Note that half-range Fourier eigenfunctions are zero at $\phi = 0$ and $\phi = \pi$.

For ϕ limited to the range

$$0 \leq \phi \leq \pi \tag{24}$$

show that 1 can be expressed as a linear combination of sin functions. That is,

$$1 = \sum_{m=1}^{\infty} a_m \sin m\phi. \tag{25}$$

Show that

$$a_m = \begin{cases} \dfrac{4}{\pi}\dfrac{1}{m}, & m = 1, 3, 5, \ldots, \\ 0, & m = 2, 4, 6, \ldots. \end{cases} \tag{26}$$

Note that the series expansion is not valid at the boundaries $\phi = 0$ and $\phi = \pi$. Evaluate the series at $\phi = \pi/2$ and find the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots. \tag{27}$$