

Homework No. 08 (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2022 Nov 14, 4.30pm

1. (20 points.) Use the integral representation of $J_m(t)$,

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it \cos \alpha - im\alpha}, \quad (1)$$

to prove the recurrence relations

$$2 \frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \quad (2a)$$

$$2 \frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t). \quad (2b)$$

2. (10 points.) Using the recurrence relations of Eq. (2), show that

$$\left(-\frac{d}{dt} + \frac{m-1}{t}\right) \left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(\frac{d}{dt} + \frac{m+1}{t}\right) \left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_m(t) \quad (3)$$

and from this derive the differential equation satisfied by $J_m(t)$.

3. (20 points.) Using the recurrence relations,

$$2 \frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \quad (4a)$$

$$2 \frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t), \quad (4b)$$

satisfied by the Bessel functions, derive the ‘ladder’ operations satisfied by the Bessel functions,

$$\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m-1}(t), \quad (5)$$

$$\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m+1}(t). \quad (6)$$

In quantum mechanics a ladder operator is a raising or lowering operator that transforms eigenfunctions by increasing or decreasing the eigenvalue.