Homework No. 11 (Fall 2022)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2022 Dec 9, 4.30pm

1. (**20 points.**) Generate 3D plots of surface spherical harmonics $Y_{lm}(\theta,\phi)$ as a function of θ and ϕ. In particular,

   (a) Plot $\text{Re}\big[Y_{73}(\theta,\phi)\big]$.

   (b) Plot $\text{Im}\big[Y_{73}(\theta,\phi)\big]$.

   (c) Plot $\text{Abs}\big[Y_{73}(\theta,\phi)\big]$.

   (d) Plot your favourite spherical harmonic, that is, choose a l and m, and Re or Im or Abs.

   Hint: In Mathematica these plots are generated using the following commands:

   ```
   SphericalPlot3D[Re[SphericalHarmonicY[l,m,θ,φ]],{θ,0,Pi},{ φ,0,2 Pi}]
   SphericalPlot3D[Im[SphericalHarmonicY[l,m,θ,φ]],{θ,0,Pi},{ φ,0,2 Pi}]
   SphericalPlot3D[Abs[SphericalHarmonicY[l,m,θ,φ]],{θ,0,Pi},{ φ,0,2 Pi}]
   ```

   Refer to diagrams in Wikipedia article on 'spherical harmonics' to see some visual representations of these functions.

2. (**20 points.**) Using the definition of spherical harmonics

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}, \tag{1}$$

   evaluate the explicit expressions for $Y_{21}(\theta,\phi)$ and $Y_{2,-2}(\theta,\phi)$.

3. (**20 points.**) Verify that the right hand side of

$$(-\mathbf{a}\cdot\boldsymbol{\nabla})\frac{1}{r} = \frac{\mathbf{a}\cdot\mathbf{r}}{r^3} \tag{2}$$

   is a solution to Laplace's equation for $\mathbf{r}\neq 0$. Further, verify the relation

$$(\mathbf{a}_1\cdot\boldsymbol{\nabla})(\mathbf{a}_2\cdot\boldsymbol{\nabla})\frac{1}{r} = \frac{1}{r^5}\Big[3(\mathbf{a}_1\cdot\mathbf{r})(\mathbf{a}_2\cdot\mathbf{r}) - (\mathbf{a}_1\cdot\mathbf{a}_2)r^2\Big], \tag{3}$$

   which is also a solution to Laplace's equation for $\mathbf{r}\neq 0$, but need not be verified here.

4. (**10 points.**) The generating function for the spherical harmonics, $Y_{lm}(\theta,\phi)$, is

$$\frac{1}{l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,Y_{lm}(\theta,\phi)\,\psi_{lm}, \tag{4}$$

where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{5}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{6}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{7}$$

and

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{8}$$

Show that

$$\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right) \tag{9}$$

is unchanged by the substitution: $y_+ \leftrightarrow y_-$, $\theta \to -\theta$, $\phi \to -\phi$. Thus, show that

$$Y_{lm}(\theta,\phi) = Y_{l,-m}(-\theta,-\phi). \tag{10}$$

5. (**20 points.**) Legendre polynomials of order l is given by (for $|t| < 1$)

$$P_l(t) = \left(\frac{d}{dt}\right)^l \frac{(t^2-1)^l}{2^l l!}. \tag{11}$$

(a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the l differentiations in Eq. (11).

(b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta). \tag{12}$$

(c) Using the orthonormality condition for the spherical harmonics

$$\int d\Omega\, Y_{lm}^*(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{13}$$

recognize the orthogonality statement for Legendre polynomials,

$$\frac{1}{2}\int_{-1}^{1} dt\, P_l(t)P_{l'}(t) = \frac{\delta_{ll'}}{2l+1}. \tag{14}$$

Use

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \tag{15}$$

to check this explicitly for $l, l' = 0, 1, 2$.