Final Exam (2023 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: 2023 May 12

1. (**20 points.**) In class.
2. (**20 points.**) In class.
3. (**20 points.**) The scattering amplitude off a scatterer of susceptibility $\chi(\mathbf{r}, \omega)$ is given by

$$f(\theta, \phi, \omega) = -\frac{k^2}{4\pi}\chi(\mathbf{k}' - \mathbf{k}, \omega), \tag{1}$$

where $\chi(\mathbf{q}, \omega)$ is the Fourier transform of $\chi(\mathbf{r}, \omega)$,

$$\chi(\mathbf{q}, \omega) = \int d^3r \, e^{i\mathbf{q}\cdot\mathbf{r}}\chi(\mathbf{r}, \omega). \tag{2}$$

If the scatterers are confined on a plane, say $z = 0$, then it is convenient to define polarizability per unit area $\boldsymbol{\lambda} = \boldsymbol{\alpha}/\text{Area}$, and the associated susceptibility

$$\boldsymbol{\chi}(\mathbf{r}, \omega) = 4\pi\boldsymbol{\lambda}(\mathbf{s})\,\delta(z), \tag{3}$$

where the δ-function has been used to describe the assumption that the obstacles in a thin film are confined to the $z = 0$ plane here. Once the scatterers are restricted on a plane, we can choose the direction of incidence of the plane wave $\mathbf{k}$ to be normal to the plane constituting the scatterers. That is, $\mathbf{k}\cdot\mathbf{s} = 0$, where $\mathbf{s}$ are the positions of the point scatterers on the plane $z = 0$. Also, note that the amplitude of the incident electric field $\mathbf{E}_0$ is independent of the position $\mathbf{s}$. Using these considerations show that the scattering amplitude, for isotropic polarizabilities, $\boldsymbol{\lambda}(\mathbf{s}) = \mathbf{1}\lambda(\mathbf{s})$, is given by

$$f(\theta, \phi, \omega) = -k^2 \int d^2s \, e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}}\lambda(\mathbf{s}). \tag{4}$$

For a disc of radius R centered at position $\mathbf{s}_0$ with uniform polarizability per unit area λ complete the integrals to obtain

$$f(\theta, \phi, \omega) = -\lambda k^2 \pi R^2 2\frac{J_1(kR\sin\theta)}{kR\sin\theta}\, e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}_0}. \tag{5}$$

Hint: Use the integral representation of zeroth order Bessel function of the first kind

$$J_0(t) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{it\cos\phi} \tag{6}$$

and the identity

$$\int_0^b tdtJ_0(t) = bJ_1(b). \tag{7}$$

Note the limiting value

$$\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2}, \tag{8}$$

which guarantees a well defined value for the scattering amplitude at $\theta = 0$. We observe the interesting feature that the scattering amplitude at $\theta = 0$ is entirely given by the area of the disc.

4. (**0 points.**) The statement of conservation of electromagnetic energy in the time domain is

$$\frac{\partial}{\partial t}U(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{S}(\mathbf{r},t) + \mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = 0, \tag{9}$$

where $U(\mathbf{r},t)$ is the electromagnetic energy density introduced by Sommerfeld and Brillouin and $\mathbf{S}(\mathbf{r},t)$ is the flux of electromagnetic energy density or the Poynting vector given by

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t). \tag{10}$$

Let us define the time average of the rate of change of electromagnetic energy density at a point as the average power density

$$p(\mathbf{r}) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{-T}^{T} dt\frac{\partial}{\partial t}U(\mathbf{r},t) = \lim_{\tau\to\infty}\frac{U(\mathbf{r},T) - U(\mathbf{r},-T)}{\tau}, \tag{11}$$

where $\tau = 2T$ is the (infinite) time for which the system evolves. Thus, we have

$$p(\mathbf{r}) + \frac{1}{\tau}\int_{-\infty}^{\infty} dt\boldsymbol{\nabla}\cdot\mathbf{S}(\mathbf{r},t) + p_{\text{abs.}}(\mathbf{r}) = 0, \tag{12}$$

where

$$p_{\text{ch.}}(\mathbf{r}) = \frac{1}{\tau}\int_{-\infty}^{\infty} dt\mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = 0. \tag{13}$$

Integrating over all space and using divergence theorem we have

$$P + \frac{1}{\tau}\int_{-\infty}^{\infty} dt \oint d\Omega r^2\hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},t) + P_{\text{abs.}} = 0. \tag{14}$$

The Poynting vector is a bilinear construction in terms of the fields. Thus, using Plancherel theorem we have

$$\int_{-\infty}^{\infty} dt\,\mathbf{S}(\mathbf{r},t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathbf{S}(\mathbf{r},\omega), \tag{15}$$

where

$$\mathbf{S}(\mathbf{r},\omega) = \frac{1}{2}\Big[\mathbf{E}(\mathbf{r},\omega)^*\times\mathbf{H}(\mathbf{r},\omega) + \mathbf{E}(\mathbf{r},\omega)\times\mathbf{H}(\mathbf{r},\omega)^*\Big], \tag{16}$$

where we introduced symmetrization under complex conjugation. This symmetrization is necessary whenever these construction appear outside the frequency integral. This symmetrization will be not written explicitly from now onwards to avoid clutter in the expressions. The total flux of electromagnetic energy, obtained by integrating over all time, in conjunction with the Plancherel theorem provides the frequency distribution of the total flux in $\mathbf{S}(\mathbf{r},\omega)$. Thus, the statement of conservation of energy in the frequency domain dictates the frequency distribution of the power to be

$$\frac{\partial P}{\partial \omega} + \frac{1}{2\pi\tau} \oint d\Omega r^2 \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r},\omega) + \frac{\partial P_{\text{abs.}}}{\partial \omega} = 0. \tag{17}$$

The decomposition

$$\mathbf{E}(\mathbf{r},\omega) = \mathbf{E}_{\text{in}}(\mathbf{r},\omega) + \mathbf{E}_s(\mathbf{r},\omega), \tag{18a}$$
$$\mathbf{B}(\mathbf{r},\omega) = \mathbf{B}_{\text{in}}(\mathbf{r},\omega) + \mathbf{B}_s(\mathbf{r},\omega), \tag{18b}$$

associated with a scattering process introduces the following decomposition in the frequency distribution of the total flux to have the form

$$\mathbf{S}(\mathbf{r},\omega) = \mathbf{S}_{\text{in}}(\mathbf{r},\omega) + \mathbf{S}_s(\mathbf{r},\omega) + \mathbf{S}_{\text{damp.}}(\mathbf{r},\omega), \tag{19}$$

where

$$\mathbf{S}_{\text{in}}(\mathbf{r},\omega) = \mathbf{E}_{\text{in}}(\mathbf{r},\omega)^* \times \mathbf{H}_{\text{in}}(\mathbf{r},\omega), \tag{20a}$$
$$\mathbf{S}_s(\mathbf{r},\omega) = \mathbf{E}_s(\mathbf{r},\omega)^* \times \mathbf{H}_s(\mathbf{r},\omega), \tag{20b}$$
$$\mathbf{S}_{\text{damp.}}(\mathbf{r},\omega) = \mathbf{E}_{\text{in}}(\mathbf{r},\omega)^* \times \mathbf{H}_s(\mathbf{r},\omega) + \mathbf{E}_s(\mathbf{r},\omega)^* \times \mathbf{H}_{\text{in}}(\mathbf{r},\omega), \tag{20c}$$

with symmetrization under complex conjugation understood implicitly. Let us define the following cross sections

$$\sigma_{\text{in}} = \frac{1}{|\mathbf{S}_{\text{in}}|} \oint d\Omega\, r^2 \hat{\mathbf{r}} \cdot \mathbf{S}_{\text{in}}(\mathbf{r},\omega), \tag{21a}$$
$$\sigma_{\text{scatt.}} = \frac{1}{|\mathbf{S}_{\text{in}}|} \oint d\Omega\, r^2 \hat{\mathbf{r}} \cdot \mathbf{S}_s(\mathbf{r},\omega), \tag{21b}$$
$$\sigma_{\text{damp.}} = \frac{1}{|\mathbf{S}_{\text{in}}|} \oint d\Omega\, r^2 \hat{\mathbf{r}} \cdot \mathbf{S}_{\text{damp.}}(\mathbf{r},\omega), \tag{21c}$$

and the total cross section

$$\sigma_{\text{tot.}} = \sigma_{\text{in}} + \sigma_{\text{scatt.}} + \sigma_{\text{damp.}} \tag{22}$$

such that the power spectrum is given by

$$\frac{\partial P}{\partial \omega} + \frac{|\mathbf{S}_{\text{in}}|}{2\pi\tau} \sigma_{\text{tot.}} + \frac{\partial P_{\text{abs.}}}{\partial \omega} = 0. \tag{23}$$

Here

$$|\mathbf{S}_{\text{in}}| = \frac{|E_0|^2}{\mu_0 c} \tag{24}$$

has the dimensions of energy per unit area times time.

In general we have

$$\sigma_{\text{in}} = 0 \tag{25}$$

because it involves an angular integration of $\hat{\mathbf{r}} \cdot \mathbf{S}_{\text{in}} \sim \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta$. In general we have

$$\sigma_{\text{scatt.}} = \oint d\Omega\, r^2 \frac{|c\mathbf{B}_s(\mathbf{r}, \omega)|^2}{|cB_0|^2}. \tag{26}$$

In terms of the scattering amplitude $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ we have the scattering cross section

$$\sigma_{\text{scatt.}} = \oint d\Omega\, \frac{|\hat{\mathbf{r}} \times \mathbf{K}(\hat{\mathbf{r}}, \omega)|^2}{|E_0|^2} = \oint d\Omega\, \frac{\Big[|\mathbf{K}|^2 - |\hat{\mathbf{r}} \cdot \mathbf{K}|^2\Big]}{|E_0|^2}. \tag{27}$$

The energy content of the scattered radiation is supplied by the energy lost by the incident beam in the dielectric material. This energy lost is given by

$$\sigma_{\text{damp.}} = -\frac{4\pi}{k} \frac{\text{Im}\Big[\mathbf{E}_0^* \cdot \mathbf{K}(\hat{\mathbf{z}}, \omega)\Big]}{|E_0|^2}. \tag{28}$$

This term is interpreted as dissipation of energy inside the dielectric medium due to the impediment of the incident beam. The cause of this dissipation is radiation damping, or reaction force experienced by the induced dipoles due to radiation. Together, we have

$$\sigma_{\text{tot.}} = \sigma_{\text{scatt.}} + \sigma_{\text{damp.}} \tag{29}$$

which is the optical theorem. In summary,

$$\frac{\partial P}{\partial \omega} + \frac{|\mathbf{S}_{\text{in}}|}{2\pi\tau}\Big[\sigma_{\text{scatt.}} + \sigma_{\text{damp.}} + \sigma_{\text{abs.}}\Big] = 0. \tag{30}$$