Midterm Exam No. 02 (2023 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: 2023 April 30

1. (**20 points.**) The electric and magnetic field generated by a particle with charge q moving along the z axis with speed v, $\beta = v/c$, can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\left[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - vt)\hat{\mathbf{k}}\right]}{(x^2 + y^2)} \frac{(x^2 + y^2)(1 - \beta^2)}{\left[(x^2 + y^2)(1 - \beta^2) + (z - vt)^2\right]^{\frac{3}{2}}}, \tag{1a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\beta} \times \mathbf{E}(\mathbf{r}, t). \tag{1b}$$

(a) Consider the distribution

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2} \frac{\epsilon}{(x^2 + \epsilon)^{\frac{3}{2}}}. \tag{2}$$

Show that

$$\delta(x) \begin{cases} \to \frac{1}{2\sqrt{\epsilon}} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon}{2x^3} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{3}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{4}$$

(b) Thus, verify that the electric and magnetic field of a charge approaching the speed of light can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\rho}}}{\rho} \delta(z - ct), \tag{5a}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \frac{2q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{\phi}}}{\rho} \delta(z - ct) = 2q \left(\frac{\mu_0 c}{4\pi}\right) \frac{\hat{\boldsymbol{\phi}}}{\rho} \delta(z - ct), \tag{5b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\rho = \sqrt{x^2 + y^2}$, $\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, and $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the associated unit vectors. These fields are confined on the $z = ct$ plane moving with speed c. Illustrate this configuration of fields using a diagram.

(c) To confirm that the above confined fields are indeed solutions to the Maxwell equations, verify the following:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon_0} q \delta^{(2)}(\boldsymbol{\rho}) \delta(z - ct), \tag{6a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{6b}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{6c}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 q c \hat{\mathbf{z}} \delta^{(2)}(\boldsymbol{\rho}) \delta(z - ct). \tag{6d}$$

This is facilitated by writing

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{7}$$

and accomplished by using the following identities:

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{8a}$$

$$\boldsymbol{\nabla}_\rho \cdot \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \boldsymbol{\nabla}_\rho \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \hat{\mathbf{z}}\, 2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{8b}$$

2. (**20 points.**) For a particle of charge q moving very close to the speed of light, $\beta \to 1$, we have the electric and magnetic fields as

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}\,\delta(z - ct), \tag{9a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z - ct), \tag{9b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$. These fields are confined on a plane perpendicular to direction of motion.

(a) In the limit $\beta \to 1$ we have

$$\boldsymbol{\beta} = \beta\,\hat{\mathbf{z}} \to \hat{\boldsymbol{\beta}}. \tag{10}$$

Show that

$$c\mathbf{B}(\mathbf{r}, t) = \hat{\boldsymbol{\beta}} \times \mathbf{E}(\mathbf{r}, t). \tag{11}$$

(b) The electromagnetic energy density involves bilinear constructions of fields. When the fields are confined to a plane, these involve bilinear δ-functions that needs to be carefully interpreted. In particular, we encounter

$$\delta(z - vt)\delta(z - vt) = \delta(z - vt)\delta(0), \tag{12}$$

where $\delta(0)$ is interpreted as the inverse of the infinitely small length L_z associated to the plane on which the fields are confined. That is,

$$\delta(0) = \lim_{L_z \to 0}\frac{1}{L_z}. \tag{13}$$

Starting from

$$U_e(\mathbf{r}, t) = \frac{\varepsilon_0}{2}E^2 \tag{14}$$

show that the contribution to the energy density from the electric field can be expressed in the form

$$\frac{\text{Energy}}{\text{Area}} = \lim_{L_z \to 0} U_e(\mathbf{r}, t)L_z = \frac{1}{2\pi}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\rho^2}\,\delta(z - ct). \tag{15}$$

Similarly, starting from

$$U_m(\mathbf{r},t) = \frac{1}{2\mu_0}B^2 \tag{16}$$

show that the energy density from the magnetic field is given by

$$\frac{\text{Energy}}{\text{Area}} = \lim_{L_z\to 0} U_m(\mathbf{r},t)L_z = \frac{1}{2\pi}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{\rho^2}\,\delta(z-ct). \tag{17}$$

Thus, show that the ratio of electric to magnetic energy density,

$$\frac{U_m(\mathbf{r},t)}{U_e(\mathbf{r},t)} = 1 \tag{18}$$

for the above configuration.

(c) Evaluate

$$\mathbf{E}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t) \tag{19}$$

for the above configuration. Show that

$$\mathbf{G} = \varepsilon_0\mathbf{E}\times\mathbf{B} = \hat{\boldsymbol{\beta}}\frac{U}{c}, \tag{20}$$

where $U = U_e + U_m$.

(d) A plane wave is characterized by $U_e = U_m$ and $\mathbf{E}\cdot\mathbf{B} = 0$. Does the above configuration satisfy the characteristics of a plane wave?