Homework No. 01 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2022 Jan 24, 4.30pm

Analytic manipulations in vector calculus are conveniently accomplished using index notation and dyadic notation. The following exercise serve as a review for the methods. Submit problems 10 and 28 for assessment. Rest are for practice.

(Algebraic) index notation

1. (**10 points.**) Verify the following relations:

$$\delta_{ij} = \delta_{ji}, \tag{1a}$$
$$\delta_{ii} = 3, \tag{1b}$$
$$\delta_{ik}\delta_{kj} = \delta_{ij}, \tag{1c}$$
$$\delta_{im}B_m = B_i, \tag{1d}$$
$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}, \tag{1e}$$
$$\varepsilon_{iik} = 0, \tag{1f}$$
$$\delta_{ij}\varepsilon_{ijk} = 0. \tag{1g}$$

2. (**10 points.**) In three dimensions the Levi-Civita symbol is given in terms of the determinant of the Kronecker δ-functions,

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \\ &\quad +\delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) \\ &\quad +\delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}).\end{aligned} \tag{2}$$

Using the above identity show that

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \tag{3a}$$
$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}, \tag{3b}$$
$$\varepsilon_{ijk}\varepsilon_{ijk} = 6. \tag{3c}$$

3. (**10 points.**) Using the property of Kronecker δ-function and Levi-Civita symbol evaluate the following using index notation.

$$\delta_{ij}\delta_{ji} = \tag{4a}$$
$$\delta_{ij}\varepsilon_{ijk} = \tag{4b}$$
$$\varepsilon_{ijm}\delta_{mn}\varepsilon_{nij} = \tag{4c}$$

4. (**20 points.**) Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol expand the left hand side of the vector equation below to express it in the form on the right hand side,

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \alpha(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) + \beta(\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}). \tag{5}$$

In particular find the numbers α and β.

5. (**20 points.**) Given

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{6}$$

where $\mathbf{B}$ is a constant (homogeneous in space) vector field. Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol in three dimensions expand the left hand side of the vector equation below to express it in the form on the right hand side,

$$\boldsymbol{\nabla} \times \mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{r}. \tag{7}$$

In particular find the numbers α and β.

6. (**10 points.**) Derive the following vector identities (using index notation)

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \tag{8}$$
$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}), \tag{9}$$

7. (**10 points.**) Use index notation or dyadic notation to show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2\mathbf{A}, \tag{10a}$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}), \tag{10b}$$
$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}. \tag{10c}$$

8. (**10 points.**) (Ref. Schwinger et al., problem 1, chapter 1.) Verify the following identities explicitly:

   (a) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2\mathbf{A}$,

   (b) $\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B})$,

   (c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$,

   (d) $\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) - (\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{B} + (\mathbf{B} \times \boldsymbol{\nabla}) \times \mathbf{A}$.

(Geometric) dyadic notation

9. (**20 points.**) Verify the following identities:

$$\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}, \tag{11a}$$

$$\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}. \tag{11b}$$

Further, show that

$$\boldsymbol{\nabla} \cdot \mathbf{r} = 3, \tag{12a}$$

$$\boldsymbol{\nabla} \times \mathbf{r} = 0. \tag{12b}$$

Here r is the magnitude of the position vector $\mathbf{r}$, and $\hat{\mathbf{r}}$ is the unit vector pointing in the direction of $\mathbf{r}$.

10. (**25 points.**) Evaluate

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3} \right), \tag{13}$$

everywhere in space, including $\mathbf{r} = 0$.
Hint: Check your answer for consistency by using divergence theorem.

11. (**10 points.**) Show that

(a) $\boldsymbol{\nabla} \frac{1}{r^n} = -\mathbf{r}\,\frac{n}{r^{n+2}}$

(b) $\boldsymbol{\nabla}\,\frac{\mathbf{r}}{r^n} = \mathbf{1}\,\frac{1}{r^n} - \mathbf{r}\,\mathbf{r}\,\frac{n}{r^{n+2}}$

(c) $\boldsymbol{\nabla} \cdot \frac{\mathbf{r}}{r^n} = \frac{(3-n)}{r^n}$

(d) $\boldsymbol{\nabla} \times \frac{\mathbf{r}}{r^n} = 0$

12. (**10 points.**) For the position vector

$$\mathbf{r} = r\,\hat{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}, \tag{14}$$

show that

$$\boldsymbol{\nabla} r = \hat{\mathbf{r}}, \quad \boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}, \quad \boldsymbol{\nabla} \cdot \mathbf{r} = 3, \quad \text{and} \quad \boldsymbol{\nabla} \times \mathbf{r} = 0. \tag{15}$$

Further, show that for $n \neq 3$

$$\boldsymbol{\nabla}\,\frac{\mathbf{r}}{r^n} = \mathbf{1}\frac{1}{r^n} - \mathbf{r}\,\mathbf{r}\frac{n}{r^{n+2}}, \quad \boldsymbol{\nabla} \cdot \frac{\mathbf{r}}{r^n} = \frac{(3-n)}{r^n}, \quad \text{and} \quad \boldsymbol{\nabla} \times \frac{\mathbf{r}}{r^n} = 0. \tag{16}$$

For $n = 3$ use divergence theorem to show that

$$\boldsymbol{\nabla} \cdot \frac{\mathbf{r}}{r^n} = 4\pi\,\delta^{(3)}(\mathbf{x}). \tag{17}$$

13. (**10 points.**) (Based on Problem 1.13, Griffiths 4th edition.) Show that

$$\boldsymbol{\nabla} r^2 = 2\mathbf{r}. \tag{18}$$

Then evaluate $\boldsymbol{\nabla} r^3$. Show that

$$\boldsymbol{\nabla}\frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2}. \tag{19}$$

Then evaluate $\boldsymbol{\nabla}(1/r^2)$.

14. (**10 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}\frac{1}{r^3} = \alpha\,\hat{\mathbf{r}}\,r^n. \tag{20}$$

Thus find α and n.

15. (**10 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{p}) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{21}$$

where $\mathbf{p}$ is a constant vector. Thus, find a and b.

16. (**20 points.**) Given

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r}) = c. \tag{22}$$

Find the scalar c.

17. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}\cdot(r^2\mathbf{r}) = a\,r^n. \tag{23}$$

Thus, find a and n.

18. (**20 points.**) Evaluate

$$\boldsymbol{\nabla}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right), \tag{24}$$

where $\mathbf{p}$ is a constant vector.

19. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}\left(\frac{1}{\mathbf{r}\cdot\mathbf{p}}\right) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{25}$$

where $\mathbf{p}$ is a constant vector. Thus, find a and b.

20. (**20 points.**) Evaluate

$$\boldsymbol{\nabla}\times\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right), \tag{26}$$

where $\mathbf{m}$ is a constant vector.

21. (**20 points.**) Given the flow velocity field

$$\mathbf{v} = \omega\rho\hat{\boldsymbol{\phi}} \tag{27}$$

determine the vorticity $\boldsymbol{\nabla} \times \mathbf{v}$ of the flow. Illustrate the flow field and the vorticity using the associated vector field lines. Here ω is a constant, and ρ and ϕ are cylindrical polar coordinates.

22. (**20 points.**) Given the flow velocity field

$$\mathbf{v} = \frac{c}{\rho}\hat{\boldsymbol{\phi}} \tag{28}$$

determine the vorticity $\boldsymbol{\nabla} \times \mathbf{v}$ of the flow. Illustrate the flow field and the vorticity using the associated vector field lines. Here c is a constant, and ρ and ϕ are cylindrical polar coordinates. Let $\rho \neq 0$.

23. (**20 points.**) The relation between the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{29}$$

For a constant (homogeneous in space) magnetic field $\mathbf{B}$, verify that

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{30}$$

is a possible vector potential by showing that Eq. (30) satisfies Eq. (29).

24. (**20 points.**) Show that

$$\boldsymbol{\nabla}(\hat{\mathbf{r}} \cdot \mathbf{a}) = -\frac{1}{r}\,\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{a}) \tag{31}$$

for a uniform (homogeneous in space) vector $\mathbf{a}$.

25. (**20 points.**) Show that

$$\boldsymbol{\nabla} \cdot \big[P_0\,\hat{\mathbf{r}}\,\theta(R - r)\big], \tag{32}$$

for a uniform (homogeneous in space) P_0, can be expressed as a sum of two terms, a surface term and a volume term. Here $\theta(x) = 1$ if $x > 0$ and 0 otherwise.

Application of δ-functions

26. (**10 points.**) A uniformly charged infinitely thin disc of radius R and total charge Q is placed on the x-y plane such that the normal vector is along the z axis and the center of the disc at the origin. Write down the charge density of the disc in terms of δ-function(s). Integrate over the charge density and verify that it returns the total charge on the disc.

27. (**10 points.**) Write down the charge density for the following configurations: Point charge, line charge, surface charge, uniformly charged disc, uniformly charged ring, uniformly charged shell, uniformly charged spherical ball.

28. (**10 points.**) The distance between two points $\mathbf{r}$ and $\mathbf{r}'$ in rectangular coordinates is explicitly given by

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}. \tag{33}$$

The charge density of a charge q at the origin is described in terms of delta functions as

$$\rho(\mathbf{r}) = q\delta(x)\delta(y)\delta(z). \tag{34}$$

Evaluate the electric potential at the observation point $\mathbf{r}$, due to a point charge q placed at source point $\mathbf{r}'$, using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{35}$$

where $\int d^3r' = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz'$. That is, evaluate the three integrals in

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{\delta(x')\delta(y')\delta(z')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}. \tag{36}$$