Homework No. 03 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2022 Feb 9, 4.30pm

1. (**Comment on Flux.**) In fluid dynamics a conservation equation (including the dissipative term) has the form
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} + s = 0, \tag{1}$$
where ρ, $\mathbf{j}$, and s, are functions of position and time. In this equation the quantity $\mathbf{j}$ is defined as the flux of the quantity ρ. That is, $\mathbf{j}$ represents the flow rate of ρ per unit area. Inadvertently, in vector calculus, the surface integral of a vector field $\mathbf{E}$ over a surface S is also defined as the flux Φ_E of the vector field,
$$\Phi_E = \int_S d\mathbf{a} \cdot \mathbf{E}. \tag{2}$$
This is confusing because the conservation equation and surface integral appear in tandem in electrodynamics. For example, Eq. (1) with $s = 0$,
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \tag{3}$$
is the conservation equation for electric charge density ρ, with the current density $\mathbf{j}$ here interpreted as the flux of charge density. However, Eq. (2) for the case of current density $\mathbf{j}$ can be written as
$$I = \int_S d\mathbf{a} \cdot \mathbf{j}, \tag{4}$$
where I is the current. Thus, current I is the flux of current density $\mathbf{j}$ (in vector calculus), while the current density $\mathbf{j}$ is the flux of charge density ρ (in fluid dynamics context).

2. (**Not for submission.**) When magnetic charges ρ_m and magnetic currents $\mathbf{j}_m$ are permitted, in addition to electric charges ρ_e and electric currents $\mathbf{j}_e$, the Maxwell equations are
$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_e, \tag{5a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = \rho_m, \tag{5b}$$
$$-\boldsymbol{\nabla} \times \mathbf{E} - \frac{\partial}{\partial t}\mathbf{B} = \mathbf{j}_m, \tag{5c}$$
$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{j}_e, \tag{5d}$$

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{6a}$$

$$\mathbf{B} = \mu_0 \mathbf{H}. \tag{6b}$$

The Lorentz force, in SI units, on an object with electric charge q_e and magnetic charge q_m is

$$\mathbf{F} = q_e \mathbf{E} + q_e \mathbf{v} \times \mathbf{B} + q_m \mathbf{H} - q_m \mathbf{v} \times \mathbf{D}. \tag{7}$$

Note the negative sign in the fourth term of the Lorentz force. This is postulated based on the observation that Maxwell equations are symmetric under the replacement

$$\rho_e \to \rho_m, \qquad \rho_m \to -\rho_e, \tag{8a}$$

$$\mathbf{j}_e \to \mathbf{j}_m, \qquad \mathbf{j}_m \to -\mathbf{j}_e, \tag{8b}$$

$$\mathbf{E} \to \mathbf{H}, \qquad \mathbf{H} \to -\mathbf{E}, \tag{8c}$$

which is a special case of duality transformation. The corresponding force density (force per unit volume) $\mathbf{f}$ is

$$\mathbf{f} = \rho_e \mathbf{E} + \mathbf{j}_e \times \mathbf{B} + \rho_m \mathbf{H} - \mathbf{j}_m \times \mathbf{D}. \tag{9}$$

The speed of light in vacuum c satisfies the relation

$$c^2 \varepsilon_0 \mu_0 = 1. \tag{10}$$

The duality transformation did entice us to consider magnetic monopoles. However, the purpose for introducing magnetic monopoles here is also to gain insight for the sources for the electromagnetic energy density and electromagnetic momentum density as suggested by the associated conservation laws deduced from the Maxwell equations. At any stage of our calculation we can get the standard electrodynamics by switching off the contributions from magnetic charges and currents, by setting $\rho_m = 0$ and $\mathbf{j}_m = 0$.

(a) Conservation of charge: Take the divergence of Ampère's law in Eq. (5d), and then use Gauss law for electric field in Eq. (5a) to deduce

$$\frac{\partial}{\partial t}\rho_e + \boldsymbol{\nabla} \cdot \mathbf{j}_e = 0. \tag{11}$$

This is the statement of conservation of electric charge. Similarly, take the divergence of Faraday's law in Eq. (5c), and then use Gauss law for magnetic field in Eq. (5b) to deduce

$$\frac{\partial}{\partial t}\rho_m + \boldsymbol{\nabla} \cdot \mathbf{j}_m = 0. \tag{12}$$

This is the statement of conservation of magnetic charge.

(b) Conservation of energy: The rate of energy transfer from the electromagnetic field to the charge, the power, is given by

$$\mathbf{F} \cdot \mathbf{v} = q_e \mathbf{v} \cdot \mathbf{E} + q_m \mathbf{v} \cdot \mathbf{H}. \tag{13}$$

For a continuous charge distribution, then, the rate of energy transfer from the electromagnetic field to charge distributions is given by

$$\mathbf{j}_e \cdot \mathbf{E} + \mathbf{j}_m \cdot \mathbf{H}. \tag{14}$$

Use Ampère's law in Eq. (5d) to replace $\mathbf{j}_e$, and Faraday's law in Eq. (5c) to replace $\mathbf{j}_m$, in Eq. (14), to obtain the statement of conservation of energy

$$\frac{\partial}{\partial t} U + \boldsymbol{\nabla} \cdot \mathbf{S} + \mathbf{j}_e \cdot \mathbf{E} + \mathbf{j}_m \cdot \mathbf{H} = 0, \tag{15}$$

where

$$U = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{H} \cdot \mathbf{B} \tag{16}$$

is the electromagnetic field energy density and

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{17}$$

is the flux of electromagnetic field energy density (the Poynting vector).
Hints: Use the identities

$$\mathbf{C} \cdot \frac{\partial}{\partial t}\mathbf{C} = \frac{\partial}{\partial t}\left(\frac{C^2}{2}\right) \tag{18}$$

for any vector $\mathbf{C}$, and

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{H} - (\boldsymbol{\nabla} \times \mathbf{H}) \cdot \mathbf{E}. \tag{19}$$

(c) Conservation of momentum: We start from the expression for the force density in Eq. (9). Use Gauss law for electric and magnetic field in Eqs. (5a) and (5b) to replace ρ_e and ρ_m, Ampère's law in Eq. (5d) to replace $\mathbf{j}_e$, and Faraday's law in Eq. (5c) to replace $\mathbf{j}_m$, in Eq. (9), to obtain the statement of conservation of momentum

$$\frac{\partial}{\partial t}\mathbf{G} + \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{f} = 0, \tag{20}$$

where

$$\mathbf{G} = \mathbf{D} \times \mathbf{B} \tag{21}$$

is the electromagnetic field momentum density and

$$\mathbf{T} = \mathbf{1}U - (\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B}) \tag{22}$$

is the flux of electromagnetic field momentum density (the stress tensor).
Hints: Use the identities

$$(\boldsymbol{\nabla} \cdot \mathbf{D})\mathbf{E} + (\boldsymbol{\nabla} \times \mathbf{E}) \times \mathbf{D} = -\frac{1}{2}\boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{D}) + \boldsymbol{\nabla} \cdot (\mathbf{E}\mathbf{D}), \tag{23a}$$

$$(\boldsymbol{\nabla} \cdot \mathbf{B})\mathbf{H} + (\boldsymbol{\nabla} \times \mathbf{H}) \times \mathbf{B} = -\frac{1}{2}\boldsymbol{\nabla}(\mathbf{H} \cdot \mathbf{B}) + \boldsymbol{\nabla} \cdot (\mathbf{H}\mathbf{B}). \tag{23b}$$

3. (**20 points.**) The electromagnetic energy density U and the corresponding energy flux vector $\mathbf{S}$ are given by, ($\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$, $\varepsilon_0 \mu_0 c^2 = 1$,)

$$U = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}), \qquad \mathbf{S} = \mathbf{E} \times \mathbf{H}. \tag{24}$$

The electromagnetic momentum density $\mathbf{G}$ and the corresponding momentum flux tensor $\mathbf{T}$ are given by

$$\mathbf{G} = \mathbf{D} \times \mathbf{B}, \qquad \mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{25}$$

Show that

$$\mathrm{Tr}(\mathbf{T}) = T_{ii} = aU \tag{26}$$

and

$$\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}) = T_{ij} T_{ji} = \alpha\, U^2 + \beta\, \mathbf{G} \cdot \mathbf{S}. \tag{27}$$

Determine a, α, and β.