

Homework No. 05 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2023 Feb 23, 4.30pm

1. (20 points.) How does the operator

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \quad (1)$$

in the wave equation transform under the Lorentz transformation

$$z' = \gamma z + \beta \gamma ct, \quad (2a)$$

$$ct' = \beta \gamma z + \gamma ct. \quad (2b)$$

2. (20 points.) Lorentz transformation describing a boost in the x -direction, y -direction, and z -direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1 \gamma_1 & 0 & 0 \\ -\beta_1 \gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2 \gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2 \gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3 \gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3 \gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \quad (3)$$

respectively. Transformation describing a rotation about the x -axis, y -axis, and z -axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \omega_1 & \sin \omega_1 \\ 0 & 0 & -\sin \omega_1 & \cos \omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega_2 & 0 & -\sin \omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin \omega_2 & 0 & \cos \omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega_3 & \sin \omega_3 & 0 \\ 0 & -\sin \omega_3 & \cos \omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4)$$

respectively. For infinitesimal transformations, $\beta_i = \delta \beta_i$ and $\omega_i = \delta \omega_i$ use the approximations

$$\gamma_i \sim 1, \quad \cos \omega_i \sim 1, \quad \sin \omega_i \sim \delta \omega_i, \quad (5)$$

to identify the generator for boosts \mathbf{N} , and the generator for rotations the angular momentum \mathbf{J} ,

$$\mathbf{L} = \mathbf{1} + \delta \boldsymbol{\beta} \cdot \mathbf{N} \quad \text{and} \quad \mathbf{R} = \mathbf{1} + \delta \boldsymbol{\omega} \cdot \mathbf{J}, \quad (6)$$

respectively. Then derive

$$[N_1, N_2] = N_1 N_2 - N_2 N_1 = J_3. \quad (7)$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

3. **(20 points.)** The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}, \quad (8)$$

where v_a and v_b are velocities and c is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [1].) An interactive applet for exploring velocity addition is available at Kocik's web page [2]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.)

- (a) Analyse what is obtained if you add two subluminal speeds?
 - (b) Analyse what is obtained if you add a subluminal speed to speed of light?
 - (c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?
 - (d) Analyse what is obtained if you add speed of light to another speed of light?
 - (e) Analyse what is obtained if you add a superluminal speed to speed of light?
 - (f) Analyse what is obtained if you add two superluminal speeds?
4. **(20 points.)** (Based on Hughston and Tod's book.) Prove the following.
- (a) If p^μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.
 - (b) If p^μ and q^μ are both time-like vectors and $p^\mu q_\mu < 0$ then either both are future-pointing or both are past-pointing.
 - (c) If p^μ and q^μ are both light-like vectors and $p^\mu q_\mu = 0$ then p^μ and q^μ are proportional.

References

- [1] J. Kocik. Geometric diagram for relativistic addition of velocities. *Am. J. Phys.*, 80:737–739, August 2012.
- [2] J. Kocik. An interactive applet for exploring relativistic velocity addition. <http://www.mathoutlet.com/2016/08/relativistic-composition-of-velocities.html>.