Homework No. 07 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2023 Mar 21, 4.30pm

1. (**20 points.**) A relativisitic particle in a uniform magnetic field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{1a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{1b}$$

where

$$E = mc^2\gamma, \tag{2a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{2b}$$

and

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{3}$$

Show that

$$\frac{d\gamma}{dt} = 0. \tag{4}$$

Then, derive

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_c, \tag{5}$$

where

$$\boldsymbol{\omega}_c = \frac{q\mathbf{B}}{m\gamma}. \tag{6}$$

Compare this relativistic motion to the associated non-relativistic motion.

2. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{7a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{7b}$$

where

$$E = mc^2\gamma, \tag{8a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{8b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{9}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{10}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{11a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{11b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{12}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{13}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{14}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{15}$$

For our configuration this implies

$$\beta_x = 0, \tag{16a}$$

$$\beta_y\gamma = \omega_0 t, \tag{16b}$$

$$\beta_z = 0. \tag{16c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1+\omega_0^2 t^2}}. \tag{17}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right]. \tag{18}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0}\right)^2 - c^2t^2 = \frac{c^2}{\omega_0^2}. \tag{19}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2t^2 = y_0^2. \tag{20}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c = \frac{c^2}{y_0}. \tag{21}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2}\alpha t^2. \tag{22}$$

Show that the hyperbolic curve

$$y = y_0\sqrt{1 + \frac{c^2 t^2}{y_0^2}} \tag{23}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{24}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2}\alpha t^2 + \dots. \tag{25}$$

3. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{26a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{26b}$$

where

$$E = mc^2\gamma, \tag{27a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{27b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{28}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{29}$$

and initial conditions

$$\mathbf{v}(0) = v_0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{30a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{30b}$$

We will use the associated definitions $\boldsymbol{\beta}_0 = \mathbf{v}(0)/c$ and $\boldsymbol{\gamma}_0 = 1/\sqrt{1-\beta_0^2}$.

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{31}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{32}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{33}$$

(b) For our configuration show that

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t + \beta_0\gamma_0\hat{\mathbf{x}}, \tag{34}$$

such that

$$\beta_x\gamma = \beta_0\gamma_0, \tag{35a}$$
$$\beta_y\gamma = \omega_0 t, \tag{35b}$$
$$\beta_z\gamma = 0. \tag{35c}$$

Using $\beta_z\gamma = 0$, learn that

$$\frac{\beta_z^2}{1-\beta_x^2-\beta_y^2-\beta_z^2} = 0 \tag{36}$$

and in conjunction with $\beta_x\gamma = \beta_0\gamma_0$ deduce that

$$\beta_z = 0 \tag{37}$$

and

$$\frac{\beta_x^2}{\beta_0^2} + \beta_y^2 = 1. \tag{38}$$

Thus, deduce

$$\gamma^2 = \omega_0^2 t^2 + \gamma_0^2 \tag{39}$$

and

$$\beta_x^2 + \beta_y^2 = \beta_0^2 + \frac{\beta_y^2}{\gamma_0^2}. \tag{40}$$

Further, deduce

$$\beta_y = \frac{\bar{\omega}_0 t}{\sqrt{1+\bar{\omega}_0^2 t^2}} \tag{41}$$

and

$$\beta_x = \frac{\beta_0}{\sqrt{1+\bar{\omega}_0^2 t^2}}, \tag{42}$$

where

$$\bar{\omega}_0 = \frac{\omega_0}{\gamma_0}. \tag{43}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right], \tag{44a}$$

$$x - x_0 = \frac{v_0}{\bar{\omega}_0}\sinh^{-1}\bar{\omega}_0 t, \tag{44b}$$

and $z = 0$.

(c) Show that for $v_0 = 0$ we reproduce the solution for a particle starting from rest. Next, for

$$\bar{\omega}_0 t \ll 1 \tag{45}$$

and

$$\alpha = \bar{\omega}_0 c \tag{46}$$

obtain the non-relativistic limits,

$$y - y_0 = \frac{1}{2}\alpha t^2, \tag{47a}$$

$$x - x_0 = v_0 t. \tag{47b}$$

Hint: Recall the series expansion

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) = x + \ldots. \tag{48}$$