Homework No. 08 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Tuesday, 2023 Mar 28, 4.30pm

1. (**20 points.**) The eigenvalues λ of the field tensor $F^{\mu\nu}/\sqrt{\mu_0}$ satisfy the quartic equation

$$\lambda^4 - 2\mathcal{L}\lambda^2 - \mathcal{G}^2 = 0. \tag{1}$$

in terms of

$$\mathcal{L} = \frac{\varepsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \quad \text{and} \quad \mathcal{G} = \varepsilon_0 \mathbf{E}\cdot c\mathbf{B}, \tag{2}$$

such that

$$-2\mu_0 c^2\mathcal{L} = c^2B^2 - E^2 \quad \text{and} \quad \mu_0 c^2\mathcal{G} = \mathbf{E}\cdot c\mathbf{B}. \tag{3}$$

(a) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{\mathcal{L} - \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}, \tag{4a}$$

$$\lambda_2 = \sqrt{\mathcal{L} + \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}. \tag{4b}$$

(b) In terms of the complex field

$$c\mathbf{X} = \frac{\mathbf{E} + ic\mathbf{B}}{\sqrt{2}} \tag{5}$$

show that

$$\mathcal{Z} = \frac{1}{\mu_0}\mathbf{X}\cdot\mathbf{X} = \mathcal{L} + i\mathcal{G} \tag{6}$$

and

$$\mathcal{Z}^* = \mathcal{L} - i\mathcal{G}. \tag{7}$$

Then, express the eigenvalues as

$$\frac{\lambda}{\sqrt{\mu_0}} = \pm\frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{L} + i\mathcal{G}} \pm \sqrt{\mathcal{L} - i\mathcal{G}}\right). \tag{8}$$

Hint: Substitute $\mathcal{Z} = Re^{i\theta}$.

(c) Show that

i. if $c^2B^2 - E^2 = 0$, then the eigenvalues are $\pm\sqrt{\mathcal{G}}$ and $\pm i\sqrt{\mathcal{G}}$.

ii. if $\mathbf{E}\cdot c\mathbf{B} = 0$, then the eigenvalues are 0, 0, and $\pm\sqrt{2\mathcal{L}}$.

(d) Which of the following is true?

i. There is no Lorentz transformation connecting two reference frames such that the field is purely magnetic in origin in one and purely electric in origin in the other.
ii. If $c^2B^2 - E^2 > 0$ in a frame, then there exists a frame in which the field is purely magnetic.
iii. If $c^2B^2 - E^2 < 0$ in a frame, then there exists a frame in which the field is purely electric.
iv. If $c^2B^2 - E^2 = 0$ in a frame, then it is so in every frame.
v. $\mathbf{E} \cdot c\mathbf{B} > 0$ in a frame, then there exists a frame in which the fields are parallel.
vi. $\mathbf{E} \cdot c\mathbf{B} < 0$ in a frame, then there exists a frame in which the fields are antiparallel.
vii. $\mathbf{E} \cdot c\mathbf{B} = 0$ in a frame, then it is so in every frame.
viii. An electromagnetic plane wave is characterized by $c^2B^2 - E^2 = 0$ and $\mathbf{E} \cdot c\mathbf{B} = 0$.