Homework No. 09 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2023 Apr 4, 4.30pm

1. (**20 points.**) Using Maxwell's equations, without introducing potentials, show that the electric and magnetic fields satisfy the inhomogeneous wave equations

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\boldsymbol{\nabla}\rho(\mathbf{r},t) - \frac{1}{\varepsilon_0}\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t), \tag{1a}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = \mu_0\boldsymbol{\nabla}\times\mathbf{J}(\mathbf{r},t). \tag{1b}$$

2. (**20 points.**) Consider the retarded Green's function

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right). \tag{2}$$

(a) For $\mathbf{r}'=0$ and $t'=0$ show that

$$G(r,t) = \frac{1}{4\pi r}\,\delta\left(t-\frac{r}{c}\right). \tag{3}$$

(b) Then, evaluate

$$\int_{-\infty}^{\infty} dt\, G(r,t). \tag{4}$$

(c) From the answer above, what can you comment on the physical interpretation of $\int_{-\infty}^{\infty} dt\, G(r,t)$.

3. (**20 points.**) Evalauate the integral

$$\int_{-\infty}^{\infty} dx\, g(x)\,\delta(b^2-a^2x^2). \tag{5}$$

Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{6}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$.

4. (**20 points.**) Evaluate the integral

$$\zeta(s) = \lim_{\epsilon \to 0+} \int_{\epsilon}^{\infty} dx \left(\frac{\pi}{x}\right)^s \delta(\sin x) \tag{7}$$

as a sum. The resultant sum is the Riemann zeta function. Determine $\zeta(2)$.
Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left| \frac{dF}{dx}\Big|_{x=a_r} \right|}, \tag{8}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$.